EFFECTIVE VISCOSITY OF A POLYDISPERSED SUSPENSION

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Abstract. We compute the first order correction of the effective viscosity for a suspension containing solid particles with arbitrary shapes. We rewrite the computation as an homogenization problem for the Stokes equations in a perforated domain. Then, we extend the method of reflections [11, 16] to approximate the solution to the Stokes problem with a fixed number of particles. By obtaining sharp estimates, we are able to prove that this method converges for small volume fraction of the solid phase whatever the number of particles. This allows to address the limit when the number of particles diverges while their radius tends to 0. We obtain a system of PDEs similar to the Stokes system with a supplementary term in the viscosity proportional to the volume fraction of the solid phase in the mixture.

1. Introduction

When a viscous fluid transports solid particles, the particles modify in return the properties of the fluid. For instance, the rheological properties of the fluid are altered. In his seminal paper [5], Einstein addresses the computation of the effective viscosity of the mixture, having in mind it could help recovering the size of the transported particles. He obtains then the formula:

$$\mu_{eff} = \mu \left(1 + \frac{5}{2}\phi + o(\phi) \right). \tag{1}$$

Here μ stands for the (bulk) viscosity of the incompressible fluid alone, μ_{eff} denotes the viscosity of the mixture and ϕ stands for the volume fraction of the solid suspension of spheres. Einstein formula has been the subject of numerous studies: analysis of Einstein "formal" computations [1, 14, 2, 9], computation of second order expansion [10, 4, 7]. We refer the reader also to [13] for a comprehensive picture on the possible phenomena influencing the effective viscosity of a suspension. Most of these studies consider homogeneous suspensions. However, as mentioned in [13], a formula for the effective viscosity depending only on the volume fraction is hopeless to describe general suspensions, the factor 5/2 in the above formula being in particular valid for a suspension of spheres *a priori*. In this paper, we provide a method for the computation of an effective viscosity allowing a distribution of shapes for the particles in the suspension.

A second motivation of the paper is to obtain a "local" formula for the effective viscosity similar to [1, 16]. To be more precise, we rephrase now the computation of an effective viscosity, as depicted in [3], into a homogenization problem. We consider an incompressible

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newtonian fluid occupying the whole space $\mathbb{R}^3$ and transporting a cloud made of N particles. We neglect the particle and fluid inertia so that computing an effective viscosity amounts to understand the behavior of the system when it is submitted to a strain flow $x \mapsto Ax$ (where A is a symmetric trace-free matrix). This reduces to the following *stationary* problem. We denote by (u,p) the fluid velocity-field/pressure. The domain of the l-th solid particle is the smooth bounded open set $B_l \subset \mathbb{R}^3$ and its center of mass is x_l. The motion of B_l is associated to a pair of translational/rotational velocities (V_l, ω_l). Introducing μ the viscosity of the fluid, the unknowns $(u, p, (V_l, \omega_l)_{l=1,\dots,N})$ are computed by solving the problem

$$\begin{cases} -\mathrm{div}\, \Sigma_\mu(u,p) = 0 \\ \mathrm{div}\, u = 0 \end{cases} \quad \text{in } \mathbb{R}^3 \setminus \bigcup_{l=1}^N \overline{B}_l, \tag{2}$$

$$\begin{cases} u(x) = V_l + \omega_l \times (x - x_l) & \text{on } \partial B_l, \text{ for } l = 1, \dots, N \\ u(x) = Ax & \text{at infinity,} \end{cases} \tag{3}$$

$$\begin{cases} \displaystyle\int_{\partial B_l} \Sigma_\mu(u,p) n \mathrm{d}s = 0 \\ \displaystyle\int_{\partial B_l} (x - x_l) \times \Sigma_\mu(u,p) n \mathrm{d}s = 0 \end{cases} \quad \text{for } l = 1, \dots, N. \tag{4}$$

In this system, we introduced the fluid stress-tensor $\Sigma_\mu(u,p)$. Under the assumption that the fluid is newtonian, it reads:

$$\Sigma_\mu(u,p) = 2\mu D(u) - p\mathbb{I}_3 = \mu(\nabla u + \nabla^\top u) - p\mathbb{I}_3.$$

The zero source terms on the right-hand side of the first equation in (2) and both equations of (4) are reminiscent of the intertialess assumption. The second equation of (3) must be understood as

$$\lim_{x \to \infty} |u(x) - Ax| = 0.$$

In the last equations (4) the symbol n stands for the normal to ∂B_l. By convention, we assume that it points inwards the solid B_l and outwards the fluid domain that we denote $\mathcal{F}_N$ in what follows:

$$\mathcal{F}_N = \mathbb{R}^3 \setminus \bigcup_{l=1}^N \overline{B}_l.$$

Under the assumption that the B_l do not overlap, existence/uniqueness of a solution to (2)-(3)-(4) falls into the scope of the classical theory for the Stokes equations (see [6, Section V]). We give a little more details in the next section. We only mention here that the pressure is unique up to a constant. But, this has no impact on our computations and we consider the pressure as being uniquely defined below (this problem could be fixed by assuming that one mean of the pressure has a fixed value). Our aim is to tackle the asymptotics of this solution when the B_l are small and many. To make this statement quantitative, we introduce further assumptions regarding the B_l. Namely, we assume that there exists a diameter $a > 0$, centers $x_l \in \mathbb{R}^3$ and shapes $\mathcal{B}_l$ (meaning smooth bounded connected open sets of $\mathbb{R}^3$) such that

$$\text{(H1)} \qquad \mathcal{B}_l \subset B(0,1), \qquad \int_{\mathcal{B}_l} x \mathrm{d}x = 0, \qquad B_l = x_l + a\mathcal{B}_l, \qquad \forall\, l = 1, \dots, N.$$

Then, we prescribe that the solid domains remain in a compact set K and that there is no-overlap between the particles:

$$\text{(H2)} \qquad B_l \subset K \quad \forall\, l = 1, \dots, N\,, \qquad d := \min_{l \neq \lambda} |x_l - x_\lambda| > 4a.$$

With these conventions, we note that the total volume of the solid phase is at most $4Na^3\pi/3$ so that, globally, in the volume K, the volume fraction of the solid phase is controlled by $4Na^3\pi/3|K|$. However, the separation assumption (H2) implies that we have also a uniform local control of the solid phase volume fraction by a^3/d^3. We use also constantly below that, with (H1)-(H2), we obtain $N \leq C|K|/d^3$.

In order to derive an effective viscosity for the mixture, the classical point of view proposed in [5, 3] is to compute the rate of work of the viscous stress tensor on the boundary ∂K of the domain K containing the solid particles:

$$W_{eff} := \int_{\partial K} \Sigma_\mu(u,p) n \cdot Ax \mathrm{d}s$$

and to compare the excess with respect to the value $W_0 = 2\mu A : A|K|$ that would yield in case there is no particle. In brief, the analysis of Einstein – in the case the B_l are spheres of radius a filling a bounded domain – relies on splitting the solution (u,p) into $u = u_0 + u_1$, $p = p_0 + p_1$. Here (u_0, p_0) is the pure strain applied on the boundaries at infinity:

$$u_0(x) = Ax \qquad p_0(x) = 0,$$

(this is a solution to the Stokes equations on $\mathbb{R}^3$ since A is trace-free) while the term (u_1, p_1) compensates the trace of the boundary conditions $u - u_0(x) = -Ax$ on the B_l that cannot be matched by a suitable pair (V_l, ω_l) in (3). Namely, one may write:

$$u(x) - u_0(x) = Ax_l - A(x - x_l) \qquad \text{on } \partial B_l.$$

Since A is symmetric the latter linear term in the boundary condition cannot be compensated by a rigid rotation. Under the assumption that the holes are well-separated one provides the approximation:

$$u_1(x) = \sum_{l=1}^N U^a[A](x - x_l), \qquad p_1(x) = \sum_{l=1}^N P^a[A](x - x_l),$$

where $(U^a[A], P^a[A])$ is the solution to the Stokes problem (2) outside $B(0,a)$ with boundary condition $U^a(x) = -Ax$ on $\partial B(0,a)$ (and vanishing boundary conditions at infinity). With this formula at-hand, one obtains that

$$W_{eff} = W_0 + \sum_{l=1}^N \int_{\partial K} \Sigma_\mu(U^a[A](\cdot - x_l), P^a[A](\cdot - x_l)) n \cdot Ax \mathrm{d}s.$$

Via conservation arguments related to the divergence form of the Stokes equation, the boundary integrals involved in W_{eff} can be transformed into N integrals over the boundaries of $B(x_l, a)$. It is then possible to apply the explicit value of the solution (U^a, P^a). Summing the contributions of all the particles leads finally to the first order expansion:

$$W_{eff} = 2\mu A : A\left(|K| + \frac{1}{2}\frac{20\pi N a^3}{3}\right),$$

leading to formula (1). We refer the reader to [3, p.246] for more details on this computation.

Herein, we show that solutions to (2)-(3)-(4) are close to solutions to the continuous analogue:

$$\begin{cases} -\mathrm{div}(2\mu[(1+\mathbb{M}_{eff})(D(u))] - p\mathbb{I}_3) = & 0 \\ \mathrm{div} u = & 0 \end{cases} \qquad \text{on } \mathbb{R}^3, \tag{5}$$

$$u(x) = Ax \qquad \text{at infinity.} \tag{6}$$

Here the symbol $(1+\mathbb{M}_{eff})(D(u))$ stands for $D(u)+\mathbb{M}_{eff}(D(u))$, with $\mathbb{M}_{eff}$ an application that maps $D(u)$ to the 3×3 matrix $\mathbb{M}_{eff}(D(u))$. This linear mapping measures the collective reaction of the particles to the strain induced by $D(u)$. We emphasize that we allow this mapping to depend on the space variable x. To be more precise, we explain now the computation of $\mathbb{M}_{eff}$. For arbitrary $l \in \{1, \dots, N\}$ let denote by $(U[A, B_l], P[A, B_l])$ the unique solution to

$$\begin{cases} -\mathrm{div}\Sigma(\mathrm{u}, \mathrm{p}) = & 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{B_l}, \\ \mathrm{div} u = & 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{B_l}, \end{cases} \qquad \begin{cases} u(x) = -Ax + V + \omega \times x & \text{on } \partial B_l, \\ u(x) = 0 & \text{at infinity ,} \end{cases} \tag{7}$$

$$\int_{\partial B_l} \Sigma(u, p) n \mathrm{d}\sigma = 0 \qquad \int_{\partial B_l} (x - x_l) \times \Sigma(u, p) n \mathrm{d}\sigma = 0. \tag{8}$$

In this system, we have $\mu = 1$ but we drop the index 1 of the symbol Σ for legibility. We note that (V, ω) are also unknowns in this problem. But they are the lagrange multipliers of the constraint (8), so that we may retain only $(U[A, B_l], P[A, B_l])$ as the solution. We associate to this solution:

$$\mathbb{M}[A, B_l] := \mathbb{P}_{3,\sigma}\left[\int_{\partial \mathcal{B}_l} -\Sigma(U[A, B_l], P[A, B_l]) n \otimes (x - x_l) + 2U[A, B_l] \otimes n \mathrm{d}s\right],$$

where $\mathbb{P}_{3,\sigma}$ stands for the orthogonal projection (w.r.t. matrix contraction) on the space of symmetric trace-free 3×3 matrices $\mathrm{Sym}_{3,\sigma}(\mathbb{R})$. As shown in Section 2 below the matrix $\mathbb{M}[A, B_l]$ encodes the far-field decay of the solution $U[A, B_l]$ in the sense that:

$$U_i[A, B_l](x) = \mathbb{M}[A, B_l] : \nabla \mathcal{U}^i(x) + l.o.t \qquad \text{for i=1,2,3} \tag{9}$$

at infinity (where $\mathcal{U}^i$ contains vector-fields build up from the Green-function for the Stokes problem). Due to the linearity of the Stokes equations, we have that, for fixed B_l the mapping $A \mapsto \mathbb{M}[A, B_l]$ is linear and thus given by a mapping $\mathbb{M}[B_l] : \mathrm{Sym}_{3,\sigma}(\mathbb{R}) \to \mathrm{Sym}_{3,\sigma}(\mathbb{R})$ (such a mapping can be identified with a 5×5 matrix). We set then:

$$\mathbb{M}_N(x) = \frac{3}{4\pi a^3} \sum_{l=1}^{N} \mathbb{M}[B_l] \mathbf{1}_{B(x_l,a)}(x) = \frac{3}{4\pi} \sum_{l=1}^{N} \mathbb{M}[\mathcal{B}_l] \mathbf{1}_{B(x_l,a)}(x) \quad \forall\, x \in \mathbb{R}^3.$$

We shall obtain below – under assumption (H1)-(H2)– that $|\mathbb{M}[\mathcal{B}_l]| \leq C$ independent of the shape $\mathcal{B}_l$. The mapping-function $\mathbb{M}_N$ has then support in K with $\|\mathbb{M}_N\|_{L^1(\mathbb{R}^3)} \lesssim a^3/d^3$ so that it is bounded independent of N. Then, one can think $\mathbb{M}_{eff}$ as a possible weak limit if the parameter N was tending to ∞.

For instance, in the case B_l are spheres of radius a (so that $\mathcal{B}_l$ is a sphere of radius 1) comparing the expansion (9) with the explicit solutions to the Stokes problem (see [8, p. 39]) we obtain that $\mathbb{M}[A, \mathcal{B}_l] = 20\pi A/3$ so that

$$\mathbb{M}_N \sim 5 \sum_{l=1}^{N} \mathbf{1}_{B(x_l,a)}.$$

In this case, the convergence of $\mathbb{M}_N$ reduces to the convergence of the distribution of centers $(x_l)_{l=1,\dots,N}$. If the empirical measures associated to the distribution of centers converges to some $f \in L^1(\mathbb{R}^3)$, we obtain, with $\phi = 4\pi N a^3/(3|K|)$ the volume fraction of particles :

$$\mathbb{M}_N \rightharpoonup 5\phi f \text{ in } L^1(\mathbb{R}^3) - w. \tag{10}$$

We give herein a quantitative result with explicit stability bounds for the distance between solutions to the perforated problem (2)-(3)-(4) and to the continuous problem (5)-(6). We restrict below to functions $\mathbb{M}_{eff}$ in classes

$$\mathcal{M}(\varepsilon) := \left\{ \mathbb{M} \in L^\infty(\mathbb{R}^3; \mathrm{Mat}_5(\mathbb{R})), \text{ s.t. } Supp(\mathbb{M}) \subset K \text{ and } \|\mathbb{M}\|_{L^\infty(K)} \leq \varepsilon \right\}.$$

Here $\varepsilon > 0$ is a given parameter related to the volume fraction a^3/d^3. We identify the space of linear mappings $\mathrm{Sym}_{3,\sigma}(\mathbb{R}) \to \mathrm{Sym}_{3,\sigma}(\mathbb{R})$ with $\mathrm{Mat}_5(\mathbb{R})$. With the notations introduced before, a precise statement of our main result is the following theorem

Theorem 1.1. *Let* (H1)-(H2) *be in force and denote by* (u_N, p_N) *the unique solution to* (2)-(3)-(4). *Let* $\varepsilon_0 > 0$, $\mathbb{M}_{eff} \in \mathcal{M}(\varepsilon_0)$ *and denote by* (u_c, p_c) *the unique solution to* (5)-(6).

Under the assumption that ε_0 *is sufficiently small and that* $a^3/d^3 < \varepsilon_0$, *for arbitary* $p \in [1, 3/2)$ *there exists a constant* C_0 *depending only on* p, ε_0, K *for which:*

$$\begin{aligned} &\|u_N - u_c\|_{L^p_{loc}(\mathbb{R}^3)} \\ &\qquad \leq C_p(K, \varepsilon_0)|A| \left[\|\mathbb{M}_N - \mathbb{M}_{eff}\|_{\dot{H}^{-1}(\mathbb{R}^3)} + \left(\frac{a^3}{d^3}\right)^{1+\theta} + \|\mathbb{M}_{eff}\|^2_{L^\infty(\mathbb{R}^3)} \right] \end{aligned} \tag{11}$$

where $\theta = \frac{1}{p} - \frac{2}{3}$.

Several comments are in order. First, In (11), the $\dot{H}^{-1}(\mathbb{R}^3)$ norm on the right-hand side must be understood componentwise. Second, in the particular case of spheres, we can compute $\mathbb{M}_{eff}$ via (10) so that, we obtain a fully rigorous justification of the system:

$$\begin{cases} -\mathrm{div}\,[2\mu\,(1+5\phi f)\,D(u) - p\mathbb{I}_3] = & 0 \\ \mathrm{div} u = & 0 \end{cases} \qquad \text{on } \mathbb{R}^3, \tag{12}$$

$$u(x) = Ax \qquad \text{at infinity,} \tag{13}$$

that has been obtained previously in [16, 1]. Finally, the restriction on exponent p is reminiscent of the singularity of solutions to (7), corresponding to the gradient of the Green function for the Stokes problem on $\mathbb{R}^3$, *i.e.* like $1/|x|^2$. This singularity allows an L^p-space for $p < 3/2$ in dimension 3. In particular, this restriction can be removed when measuring the distance between u_N and u_c outside the particle domain K (see [16] in the case of spheres).

As in the original proof of Einstein, Theorem 1.1 relies on two main properties. First, each particle in the cloud behaves as if it was alone in the strain flow $x \mapsto Ax$. Second, there is an underlying additivity principle which implies that the action of the cloud of particles on the fluid is the sum of the undividual actions of the different particles. In the two next sections, we justify the first of these two properties by extending Einstein computations to general suspensions. Broadly, a first guess (u,p) for a solution to (2)-(3)-(4) could be

$$u^{(0)}_{app}(x) = Ax \qquad p^{(0)}_{app}(x) = 0.$$

This yields a solution to (2) and (4) which does not fulfill the boundary conditions (3) on ∂B_l. So, we apply the linearity of the Stokes problem and introduce a first corrector:

$$\begin{cases} u_1(x) = & \displaystyle\sum_{l=1}^N U[A,B_l](x-x_l) \\ p_1(x) = & \displaystyle\sum_{l=1}^N \mu P[A,B_l](x-x_l) \end{cases} \qquad \text{where} \begin{cases} U[A,B_l](x) = & aU[A,\mathcal{B}_l]\left(\dfrac{x}{a}\right) \\ P[A,B_l](x) = & P[A,\mathcal{B}_l]\left(\dfrac{x}{a}\right) \end{cases}$$

Again the candidate $u^{(1)}_{app} = u^{(0)}_{app} + u_1$ is a solution to (2) and (4) but does not match boundary conditions (3). So, we proceed with compensating again the non rigid part of the velocity-field $u^{(1)}_{app}$ on the boundaries ∂B_l. This starts a process known as the "method of reflections". It has been studied in other contexts in [11, 15, 12] and extended to the problem of effective viscosity for a suspension of spheres in [16]. Herein, we modify a bit the method by correcting only the first order term in the expansion of the boundary values of $u^{(k)}_{app}$ on ∂B_l :

$$u^{(k)}_{app}(x) = V^{(k)}_l + \tilde{A}^{(k)}_l \cdot (x-x_l) + O(|x-x_l|^2), \qquad V^{(k)}_l = u^{(k)}_{app}(x_l),\ \tilde{A}^{(k)}_l = \nabla u^{(k)}_{app}(x_l).$$

This enables to rely on the semi-explicit solutions $(U[A,\mathcal{B}_l],P[A,\mathcal{B}_l])$ to (7) and relate the final computations with the associated $\mathbb{M}[A,B_l]$. However, this does not rule out the key-difficulty of the process. Indeed, the method of reflections leads to the iterative formula:

$$A_l^{(k+1)} = \sum_{\lambda\neq l} D(U)[A_\lambda^{(k)},B_\lambda](x_l - x_\lambda).$$

with a kernel $D(U)[A,B_\lambda]$ wich decays generically like $x\mapsto a^3A/|x|^3$. A priori, the above iterative formula entails then the bound:

$$\max_l |A_l^{(k+1)}| \lesssim \left(\frac{a^3}{d^3}\right)|\ln(N)| \max_l |A_l^{(k)}|$$

which yields that a^3/d^3 must be small w.r.t. $\ln(N)$ for the method to converge (see assumption (2.3) in [16]). We remove this difficulty herein by showing that there exists a Calderòn-Zygmund operator underlying the above recursive formula. This enables to rule out the limitation on a^3/d^3 with respect to the number N of particles. These computations are explained in the two next sections. Section 2 is devoted to the analysis of the problem (7)-(8). The Section 3 builds up on this analysis to study the convergence of the method of reflections and compute error estimates between the sequence of approximated solutions $u_{app}^{(n)}$ and the exact solution u_N to (2)-(3)-(4).

The two last sections are devoted to the proof of the additivity principle and to complete the proof of Theorem 1.1. Once the method of reflections is proved to converge, we have an expansion of the solution to (2)-(3)-(4) in terms of the parameter a^3/d^3. We prove that there exits an equivalent expansion of the solution to (5)-(6) w.r.t. $\mathbb{M}_{eff}$ so that there is a correspondance between the first terms in the expansions of both solutions. We emphasize that, as classical with the weak-formulation of the Stokes problem, one obtains estimates on the difference of velocity-fields $u_N - u_c$. Regularity properties of the Stokes problem entail then similar properties for the pressures.

Through the paper, we use the following conventions. In the space of 3×3 matrices $\mathrm{Mat}_3(\mathbb{R})$, we denote by $\mathrm{Sym}_3(\mathbb{R})$ the set of symmetric matrices and $\mathrm{Sym}_{3,\sigma}(\mathbb{R})$ its subspace containing only the trace-free ones. We denote $\mathbb{P}_{3,\sigma}$ the orthogonal projection from $\mathrm{Mat}_3(\mathbb{R})$ onto $\mathrm{Sym}_{3,\sigma}(\mathbb{R})$ with respect to the matrix contraction.

Concerning function spaces, we use classical notations for Lebesgue and Sobolev spaces. We also introduce the Beppo-Levi space $\dot{H}^1$ and its divergence-free variant:

$$\dot{H}^1_\sigma(\mathcal{O}) := \{u \in \dot{H}^1(\mathcal{O}) \text{ such that } \mathrm{div} u = 0 \text{ on } \mathcal{O}\}.$$

In the whole paper, we denote $\mathbf{U} := (U^{ij})_{1\leq i,j\leq 3}$ and $\mathbf{q} := (q_1,q_2,q_3)$ the fundamental solution to the Stokes equation in the whole space $\mathbb{R}^3$, which can be written

$$U^{ij} := -\frac{1}{8\pi}\left[\frac{\delta_{ij}}{|x-y|} + \frac{(x_i-y_i)(x_j-y_j)}{|x-y|^3}\right], \qquad q_j = \frac{1}{4\pi}\frac{x_j-y_j}{|x-y|^3}.$$

for $i,j = 1,2,3$. We collect $(U^{i,1},U^{i,2},U^{i,3})$ in the vector $\mathcal{U}^i$.

We also introduce the Bogovskii operator $\mathfrak{B}_{\mathcal{B}}[f]$ defined for arbitrary mean-free $f\in L^2(\mathcal{B})$. It is well-known that this $\mathfrak{B}_{\mathcal{B}}$ is continuous with values in $H^1_0(\mathcal{B})$ and characterized by $\mathrm{div}\mathcal{B}_{\mathcal{B}}[f] = f$ in $\mathcal{B}$. In particular, we denote $\mathrm{div}\mathfrak{B}_{\lambda_1,\lambda_2}[f] := \mathfrak{B}_{B(0,\lambda_2)\setminus\overline{B(0,\lambda_1)}}[f]$ for any $0 < \lambda_1 < \lambda_2$.

2. Analysis of the Stokes problem

In the whole section, we suppose that $\mathcal{B} \subset B(0,1) \subset \mathbb{R}^3$ has smooth boundaries $\partial\mathcal{B}$. Given a trace-free $A \in \mathrm{Sym}_{3,\sigma}(\mathbb{R})$, let consider the following problem:

$$\begin{cases} -\mathrm{div}\Sigma(\mathrm{u},\mathrm{p}) = 0 & \text{in } \mathbb{R}^3 \setminus \overline{\mathcal{B}}, \\ \mathrm{div}u = 0 & \text{in } \mathbb{R}^3 \setminus \overline{\mathcal{B}}, \end{cases} \qquad \begin{cases} u(x) = -Ax + V + \omega \times x & \text{on } \partial\mathcal{B}, \\ u(x) = 0 & \text{at infinity }, \end{cases} \tag{14}$$

$$\int_{\partial\mathcal{B}} \Sigma(u,p)n\mathrm{d}\sigma = 0 \qquad \int_{\partial\mathcal{B}} x \times \Sigma(u,p)n\mathrm{d}\sigma = 0. \tag{15}$$

It is classical that, given alternatively a 3×3 matrix A and $V,\omega \in \mathbb{R}^3 \times \mathbb{R}^3$, there exists a unique solution $(u[A],p[A])$ (with $V = \omega = 0$) and $(u[V,\omega],p[V,\omega])$ (with $A = 0$) to (14) in $\dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. The mapping

$$(V,\omega) \mapsto (\int_{\partial\mathcal{B}} \Sigma(u[V,\omega],p[V,\omega])n\mathrm{d}\sigma, \int_{\partial\mathcal{B}} x \times \Sigma(u[V,\omega],p[V,\omega])n\mathrm{d}\sigma)$$

is then linear and symmetric positive definite. In particular, there exists a unique solution (V_A,ω_A) to the problem:

$$\begin{aligned} \int_{\partial\mathcal{B}} \Sigma(u[V_A,\omega_A],p[V_A,\omega_A])n\mathrm{d}\sigma &= -\int_{\partial\mathcal{B}} \Sigma(u[A],p[A])n\mathrm{d}\sigma \\ \int_{\partial\mathcal{B}} x \times \Sigma(u[V_A,\omega_A],p[V_A,\omega_A])n\mathrm{d}\sigma &= -\int_{\partial\mathcal{B}} x \times \Sigma(u[A],p[A])n\mathrm{d}\sigma. \end{aligned}$$

The candidate $U[A,\mathcal{B}] = u[A] + u[V_A,\omega_A]$, $P[A,\mathcal{B}] = p[A] + p[V_A,\omega_A]$ is then a solution to (14)-(15). By difference and integration by parts, we obtain uniqueness of a velocity-field solution which enables to recover that the pressure is unique up to a constant also. Since $U[A,\mathcal{B}]$ matches a velocity-field of the form $-Ax + V + \omega \times x$ on $\partial\mathcal{B}$, it is classical to extend $U[A,\mathcal{B}]$ by the field corresponding to this boundary value on $\mathcal{B}$ yielding a vector-field $U[A,\mathcal{B}] \in \dot{H}^1_\sigma(\mathbb{R}^3)$. Straightforward integration by parts arguments show that this extended $U[A,\mathcal{B}]$ realizes:

$$\min\Big\{\int_{\mathbb{R}^3} |D(u)|^2, \quad u \in \dot{H}^1_\sigma(\mathbb{R}^3), \quad D(u) = -A \text{ on } \mathcal{B}\Big\}.$$

In particular, we note that the set on which the minimum is computed on the right-hand side increases when $\mathcal{B}$ decreases. Since we assume $\mathcal{B} \subset B(0,1)$ in this section, we infer a uniform bound for $\|D(U[A,\mathcal{B}])\|_{L^2(\mathbb{R}^3)}$ by the minimum reached for $\mathcal{B} = B(0,1)$. This yields that

$$\|D(U[A,\mathcal{B}])\|_{L^2(\mathbb{R}^3)} \leq C|A| \tag{16}$$

(and thus $\|U[A,\mathcal{B}]\|_{\dot{H}^1(\mathbb{R}^3)} \leq C|A|$ also) with a constant C uniform in $\mathcal{B} \subset B(0,1)$.

One may proceed similarly to show that, under the assumption (H1)-(H2), the problem (2)-(3)-(4) admits a unique solution (u_N, p_N) such that

$$v_N : x \mapsto u_N(x) - Ax \in \dot{H}^1(\mathcal{F}_N), \qquad x \mapsto p_N(x) \in L^2(\mathcal{F}_N) \tag{17}$$

Furthermore, the velocity-field of this solution can be extended to the whole $\mathbb{R}^3$ to yield a vector-field $v_N(x) = u_N(x) - Ax$ that realizes:

$$\min\Big\{ \int_{\mathbb{R}^3} |D(u)|^2, \quad u \in \dot{H}^1_\sigma(\mathbb{R}^3), \quad D(u) = -A \text{ on } \mathcal{B}_l \quad \forall\, l = 1, \dots, N \Big\}.$$

In particular, under assumptions (H1)-(H2) we can construct an extension $\tilde{v}_N$ on $\bigcup_l B(x_l, 2a)$ of the field that matches $x \mapsto -A(x - x_l)$ on each of the $B(x_l, a)$ (and thus on $B_l \subset B(x_l, a)$) by truncating and lifting the divergence terms. Straightforward computations show that we have then:

$$\int_{\mathbb{R}^3} |\nabla v_N|^2 = 2 \int_{\mathbb{R}^3} |D(v_N)|^2 \leq 2 \int_{\mathbb{R}^3} |D(\tilde{v}_N)|^2 \leq C \frac{a^3}{d^3} |A|.$$

so that there exists a uniform constant C for which:

$$\|\nabla v_N\|_{L^2(\mathbb{R}^3)} \leq C \left(\frac{a^3}{d^3} \right)^{\frac{1}{2}} . \tag{18}$$

Before going to the main result of this section, we prepare the proof with a control on momenta of the trace of

$$\Sigma[A, \mathcal{B}] = 2D(U)[A, \mathcal{B}] - P[A, \mathcal{B}]\mathbb{I}_3.$$

on $\partial\mathcal{B}$. This is the content of the following preliminary lemma:

Lemma 2.1. *There exists an absolute constant C such that:*

$$\left| \mathbb{P}_{3,\sigma} \left[\int_{\partial\mathcal{B}} \Sigma[A, \mathcal{B}] n \otimes y d\sigma(y) \right] \right| \leq C|A|,$$

We recall that $\mathbb{P}_{3,\sigma}$ stands for the orthogonal projection from $\mathrm{Mat}_3(\mathbb{R})$ onto $\mathrm{Sym}_{3,\sigma}(\mathbb{R})$. With this lemma, we obtain that the linear mappings

$$A \mapsto \mathbb{P}_{3,\sigma} \left[\int_{\partial\mathcal{B}} \Sigma[A, \mathcal{B}] n \otimes y d\sigma(y) \right]$$

are uniformly bounded whatever $\mathcal{B} \subset B(0,1)$.

Proof. Because of the linearity of the Stokes equations and of the stress tensor, the mapping from $\mathrm{Sym}_{3,\sigma}(\mathbb{R})$ to $\mathrm{Sym}_{3,\sigma}(\mathbb{R})$:

$$\mathcal{L} : \; A \mapsto \mathbb{P}_{3,\sigma} \left[\int_{\partial\mathcal{B}} \Sigma[A, \mathcal{B}] n \otimes y d\sigma(y) \right]$$

is also linear. So, let $(\mathcal{E}_i)_{i=1,\dots,5}$ an orthonormal basis of $\mathrm{Sym}_{3,\sigma}(\mathbb{R})$, and $(V_i := U[\mathcal{E}_i, \mathcal{B}])_{i=1,\dots,5}$ the corresponding velocity-fields solution to the Stokes problem (14)-(15). Then, the mapping $\mathcal{L}$ is represented in this basis by the matrix $\mathbb{L}$:

$$\mathbb{L} := \Big(\int_{\partial\mathcal{B}} \Sigma[\mathcal{E}_i, \mathcal{B}] n \otimes y d\sigma(y) : \mathcal{E}_j \Big)_{1\leq i,j\leq 5},$$

Our proof reduces to obtaining that $|\mathbb{L}_{i,j}| \leq C$ for arbitrary i, j in $\{1, \dots, 5\}$. So, let fix i, j, by integrating by parts, we have that

$$\mathbb{L}_{i,j} = \int_{\partial\mathcal{B}} \Sigma[\mathcal{E}_i, \mathcal{B}] n \cdot (\mathcal{E}_j y) \mathrm{d}\sigma(y) = 2 \int_{\mathbb{R}^3\setminus\overline{\mathcal{B}}} D(V_i) : D(V_j).$$

so that:

$$|\mathbb{L}_{i,j}| \leq 2\|D(V_i)\|_{L^2(\mathbb{R}^3\setminus\overline{\mathcal{B}})} \|D(V_j)\|_{L^2(\mathbb{R}^3\setminus\mathcal{B})} \leq C|\mathcal{E}_i||\mathcal{E}_j|,$$

where we applied (16) to obtain the last inequality. This concludes the proof. □

We continue the analysis of (14)-(15) by providing pointwise estimates on $U[A, \mathcal{B}]$. The content of the following theorem is reminiscent of [6, Section V.3]:

Proposition 2.2. *Let* $(U[A, \mathcal{B}], P[A, \mathcal{B}])$ *be the unique solution to* (14)-(15). *There exists a vector field* $\mathcal{H}[A, \mathcal{B}]$ *depending on* A *and* $\mathcal{B}$*, such that for any* $|x| > 4$,

$$U[A, \mathcal{B}](x) = \mathcal{K}[A, \mathcal{B}](x) + \mathcal{H}[A, \mathcal{B}](x),$$

where $\mathcal{K}[A, \mathcal{B}]_i(x) = \mathbb{M}[A, \mathcal{B}] : \nabla \mathcal{U}^i(x)$ *for* $i = 1, 2, 3$ *with:*

$$\mathbb{M}[A, \mathcal{B}] = \mathbb{P}_{3,\sigma} \left\{ \int_{\partial\mathcal{B}} [-(\Sigma[A, \mathcal{B}] n) \otimes y + 2U[A, \mathcal{B}] \otimes n] \, d\sigma(y) \right\}. \tag{19}$$

Moreover, there exists a constant C *independent of* A *for which:*

$$|\mathbb{M}[A, \mathcal{B}]| \leq C|A| , \qquad |\nabla^{\beta} \mathcal{H}[A, \mathcal{B}](x)| \leq C|A| \frac{1}{|x|^{3+|\beta|}} \quad \forall \beta \in \mathbb{N}^3.$$

Before giving a proof of this proposition, we note that for large x, we have:

$$\nabla \mathcal{U}^i(x) \sim \frac{C}{|x|^2}.$$

Consequently the splitting that we obtain in the above proposition corresponds to the extraction of the leading order term $(\mathcal{K}[A, \mathcal{B}](x))$ at infinity. A second crucial remark induced by this proposition is that the amplitude of both terms (the leading term $\mathcal{K}$ and remainder $\mathcal{H}$) do not depend asymptotically on the shape $\mathcal{B}$.

Proof. Let $\chi(x) \in C_0^{\infty}(\mathbb{R}^3)$ such that $\chi \equiv 1$ in $\mathbb{R}^3 \setminus B(0, 2)$ and $\chi \equiv 0$ in $B(0, 1)$. We recall that $\mathfrak{B}_{\lambda_1,\lambda_2}$ stands for the Bogovskii operator lifting the divergence on the annulus $B(0, \lambda_2) \setminus \overline{B(0, \lambda_1)}$.

By standard ellipticity arguments $U[A,\mathcal{B}], P[A,\mathcal{B}]$ are C^∞ on $\mathbb{R}^3 \setminus \overline{\mathcal{B}}$. Let define

$$\begin{aligned}\bar{U}[A,\mathcal{B}](x) &:= U[A,\mathcal{B}](x)\chi(x) - \mathfrak{B}_{1,2}[U[A,\mathcal{B}]\cdot\nabla\chi](x)\\ \bar{P}[A,\mathcal{B}](x) &:= P[A,\mathcal{B}](x)\chi(x).\end{aligned}$$

Up to a mollifying argument that we skip for conciseness, we may assume that $\bar{U}[A,\mathcal{B}] \in C^\infty(\mathbb{R}^3)$. The pair $(\bar{U}[A,\mathcal{B}], \bar{P}[A,\mathcal{B}])$ satisfies then the Stokes equation on $\mathbb{R}^3$ with source term $f_\chi[A,\mathcal{B}] = -\mathrm{div}\,\bar{\Sigma}[A,\mathcal{B}] \in C^\infty_c(B(0,2)\setminus\overline{B(0,1)})$ where:

$$\bar{\Sigma}[A,\mathcal{B}] := -\bar{P}[A,\mathcal{B}]\mathbb{I}_3 + 2D(\bar{U}[A,\mathcal{B}]).$$

Since $\bar{U}[A,\mathcal{B}] \in \dot{H}^1(\mathbb{R}^3)$ and we have uniqueness of $\dot{H}^1$-solutions to the Stokes equations on $\mathbb{R}^3$, we may use the Green function $\mathbf{U}$ to compute $\bar{U}[A,\mathcal{B}]$. This entails that, for each $i = 1,2,3$, we have:

$$\bar{U}[A,\mathcal{B}]_i = \int_{\mathbb{R}^3} f_\chi[A,\mathcal{B}](y)\cdot\mathcal{U}^i(x-y)dy.$$

In particular, for $|x| > 2$ and $i = 1,2,3$ (where $\bar{U}[A,\mathcal{B}]$ coincides with $U[A,\mathcal{B}]$), a Taylor expansion yields:

$$\begin{aligned}U[A,\mathcal{B}]_i(x) &= \int_{\mathbb{R}^3} f_\chi[A,\mathcal{B}](y)dy\cdot\mathcal{U}^i(x) - \int_{\mathbb{R}^3} f_\chi[A,\mathcal{B}](y)\otimes y dy : \nabla\mathcal{U}^i(x)\\ &\quad + \sum_{|\alpha|=2}\int_{\mathbb{R}^3} f_\chi[A,\mathcal{B}](y)\cdot\int_0^1 (1-t)y^\alpha D^\alpha\mathcal{U}^i(x-ty)dtdy.\\ &= T_0 + \mathcal{K}[A,\mathcal{B}](x) + \mathcal{H}[A,\mathcal{B}](x).\end{aligned}$$

Concerning T_0, we notice that

$$\begin{aligned}\int_{\mathbb{R}^3} f_\chi[A,\mathcal{B}](y)dy &= \int_{B(0,3)\setminus B(0,1)} \mathrm{div}\bar{\Sigma}[A,\mathcal{B}]\\ &= \int_{\partial B(0,3)} \bar{\Sigma}[A,\mathcal{B}](y)nd\sigma(y) = \int_{\partial B(0,3)} \Sigma[A,\mathcal{B}](y)nd\sigma(y)\\ &= \int_{\partial\mathcal{B}} \Sigma[A,\mathcal{B}](y)nd\sigma(y) = 0\end{aligned}$$

Hence $T_0 = 0$. To analyse $\mathcal{K}[A,\mathcal{B}](x)$, we denote:

$$\mathcal{M}[A,\mathcal{B}] := \frac{1}{2}\Big(\int_{\mathbb{R}^3} f_\chi[A,\mathcal{B}](y)\otimes y dy + (\int_{\mathbb{R}^3} f_\chi[A,\mathcal{B}](y)\otimes y dy)^T\Big)$$

and

$$\mathcal{A}[A,\mathcal{B}] := \frac{1}{2}\Big(\int_{\mathbb{R}^3} f_\chi[A,\mathcal{B}](y)\otimes y dy - (\int_{\mathbb{R}^3} f_\chi[A,\mathcal{B}](y)\otimes y dy)^T\Big).$$

First, for arbitrary skew-symmetric matrix E, there holds:

$$\begin{aligned}\int_{\mathbb{R}^3} f_\chi[A,\mathcal{B}](y)\cdot(Ey)dy &= \int_{B(0,3)\setminus B(0,1)} f_\chi[A,\mathcal{B}](y)\cdot(Ey)dy\\ &= \int_{\partial B(0,3)}\left[\bar{\Sigma}[A,\mathcal{B}]n\right]\cdot Eyd\sigma(y) - \int_{B(0,3)\setminus B(0,1)}\bar{\Sigma}[A,\mathcal{B}]:\nabla(Ey)dy,\end{aligned}$$

On the right-hand side, we have, since $\bar{\Sigma}[A,\mathcal{B}]$ is symmetric and E is skew-symmetric:

$$\int_{B(0,3)\setminus B(0,1)}\bar{\Sigma}[A,\mathcal{B}]:\nabla(Ey)dy = \int_{B(0,3)\setminus B(0,1)}\bar{\Sigma}[A,\mathcal{B}]:E\,dy = 0.$$

We also notice that

$$\begin{aligned}\int_{\partial B(0,3)}\left[\bar{\Sigma}[A,\mathcal{B}]n\right]\cdot Eyd\sigma(y) &= \int_{\partial B(0,3)}\left[\Sigma[A,\mathcal{B}]n\right]\cdot Eyd\sigma(y)\\ &= \int_{B(0,3)\setminus B(0,1)}\Sigma[A,\mathcal{B}]:\nabla(Ey)dy - \int_{\partial\mathcal{B}}\left[\Sigma[A,\mathcal{B}]n\right]\cdot Eyd\sigma(y)\\ &= -\int_{\partial\mathcal{B}}\left[\Sigma[A,\mathcal{B}]n\right]\cdot Eyd\sigma(y) = 0.\end{aligned}$$

To obtain the last equality, we use that since E is skew-symmetric, there is a vector e such that $Ex = e\times x$ and:

$$\int_{\partial\mathcal{B}}\left[\Sigma[A,\mathcal{B}]n\right]\cdot Eyd\sigma(y) = \int_{\partial\mathcal{B}}\Sigma[A,\mathcal{B}]n\times yd\sigma(y)\cdot e = 0.$$

Therefore we obtain that $\mathcal{A}[A,\mathcal{B}] = 0$. Consequently, we have

$$\mathcal{K}[A,\mathcal{B}](x) = \mathcal{M}[A,\mathcal{B}]:\nabla\mathcal{U}^i(x).$$

Since $\mathcal{M}[A,\mathcal{B}]$ is symmetric, we deduce that:

$$\mathcal{K}[A,\mathcal{B}](x) = \mathcal{M}[A,\mathcal{B}]:D(\mathcal{U}^i)(x) = \mathbb{P}_{3,\sigma}[\mathcal{M}[A,\mathcal{B}]]:D(\mathcal{U}^i)(x) = \mathbb{P}_{3,\sigma}[\mathcal{M}[A,\mathcal{B}]]:\nabla\mathcal{U}^i(x).$$

So, we set $\mathbb{M}[A,\mathcal{B}] = \mathbb{P}_{3,\sigma}[\mathcal{M}[A,\mathcal{B}]]$ and we turn to show (19) and $|\mathbb{M}[A,\mathcal{B}]|\leq C|A|$. To this end, we notice that $\mathbb{M}[A,\mathcal{B}]$ is completely fixed by its action on matrices $S\in \mathrm{Sym}_{3,\sigma}(\mathbb{R})$. So, let fix $S\in\mathrm{Sym}_{3,\sigma}(\mathbb{R})$. We have

$$\int_{\mathbb{R}^3} f_\chi[A,\mathcal{B}](y)\otimes ydy:S = \int_{\mathbb{R}^3} f_\chi[A,\mathcal{B}](y)\cdot(Sy)dy = \int_{B(0,3)\setminus\mathcal{B}} f_\chi[A,\mathcal{B}](y)\cdot(Sy)dy.$$

Applying that $\mathrm{div}\bar{\Sigma}[A,\mathcal{B}] = f_\chi[A,\mathcal{B}]$ again, we obtain

$$\begin{aligned}\int_{B(0,3)\setminus\mathcal{B}} f_\chi[A,\mathcal{B}](y)\cdot(Sy)dy &= \int_{\partial B(0,3)}(\bar{\Sigma}[A,\mathcal{B}]n)\cdot(Sy)d\sigma(y) - \int_{B(0,3)\setminus\mathcal{B}}\bar{\Sigma}[A,\mathcal{B}]:Sdy\\ &= \int_{\partial B(0,3)}(\bar{\Sigma}[A,\mathcal{B}]n)\cdot(Sy)d\sigma(y) - 2\int_{B(0,3)\setminus\mathcal{B}} D(\bar{U}[A,\mathcal{B}]):Sdy\\ &\quad+\int_{B(0,3)\setminus\mathcal{B}}\bar{P}[A,\mathcal{B}]\mathbb{I}_3:Sdy\end{aligned}$$

Since S is trace-free, the last pressure term vanishes. We rewrite the first term on the right-hand side:

$$\begin{aligned}\int_{\partial B(0,3)} (\bar{\Sigma}[A,\mathcal{B}]n)\cdot(Sy)d\sigma(y) &= \int_{\partial B(0,3)} (\Sigma[A,\mathcal{B}]n)\cdot(Sy)d\sigma(y)\\ &= -\int_{\partial\mathcal{B}} (\Sigma[A,\mathcal{B}]n)\cdot(Sy)d\sigma(y) + 2\int_{B(0,3)\setminus\mathcal{B}} D(U[A,\mathcal{B}]) : Sdy\end{aligned}$$

This entails that:

$$\begin{aligned}\int_{B(0,3)\setminus\mathcal{B}} f_\chi[A,\mathcal{B}](y)\cdot(Sy)dy &= -\int_{\partial\mathcal{B}} (\Sigma[A,\mathcal{B}]n)\cdot(Sy)d\sigma(y)\\ &\quad +2\int_{B(0,3)\setminus\mathcal{B}} D(U[A,\mathcal{B}]-\bar{U}[A,\mathcal{B}]) : Sdy.\end{aligned}$$

We recall that pressure term do vanish since S is trace-free. Concerning the first integral on the right-hand side, we notice again that:

$$\int_{\partial\mathcal{B}} (\Sigma[A,\mathcal{B}]n)\cdot(Sy)d\sigma(y) = \mathbb{P}_{3,\sigma}\left[\int_{\partial\mathcal{B}} (\Sigma[A,\mathcal{B}]n)\otimes y d\sigma(y)\right] : S.$$

We are then in position to apply Lemma 2.1 which yields that

$$\left|\int_{\partial\mathcal{B}} (\Sigma[A,\mathcal{B}]n)\cdot(Sy)d\sigma(y)\right| \leq C|A||S|, \tag{20}$$

for an absolute constant C. We proceed with the second integral. We notice that $U[A,\mathcal{B}](x) = \bar{U}[A,\mathcal{B}](x)$ for any $|x|>2$ and $\bar{U}[A,\mathcal{B}](x) = 0$ on $\partial\mathcal{B}$. Hence

$$\begin{aligned}\int_{B(0,3)\setminus\mathcal{B}} D(U[A,\mathcal{B}]-\bar{U}[A,\mathcal{B}]) : Sdy &= \int_{B(0,3)\setminus\mathcal{B}} D\big[U[A,\mathcal{B}]-\bar{U}[A,\mathcal{B}]\big]dy : S\\ &= \int_{\partial B(0,3)\cup\partial\mathcal{B}} (U[A,\mathcal{B}]-\bar{U}[A,\mathcal{B}])\otimes n d\sigma : S\\ &= \int_{\partial\mathcal{B}} U[A,\mathcal{B}]\otimes n d\sigma : S\\ &= A : S\ |\mathcal{B}|,\end{aligned}$$

where we applied that $U[A,\mathcal{B}](y) = Ay + V + \omega\times y$ to obtain the last identity. Gathering the previous computations we obtain that, for arbitrary $S\in \mathrm{Sym}_{3,\sigma}(\mathbb{R})$, there holds:

$$\begin{aligned}\mathbb{M}[A,\mathcal{B}] : S &= \int_{\mathbb{R}^3} f_\chi[A,\mathcal{B}](y)\cdot Sydy\\ &= -\mathbb{P}_{3,\sigma}\left[\int_{\partial\mathcal{B}} \Sigma[A,\mathcal{B}]n\otimes y d\sigma(y)\right] : S + 2\int_{\partial\mathcal{B}} U[A,\mathcal{B}](y)\otimes n d\sigma(y) : S.\end{aligned}$$

This concludes the proof of (19). Recalling that $\mathcal{B}\subset B(0,1)$, and applying the explicit computation of the last integral in this latter identity, we obtain also $|\mathbb{M}[A,\mathcal{B}]|\leq C|A|$.

To finish the proof, we handle the last term $\mathcal{H}[A,\mathcal{B}](x)$ for $|x|>4$. We prove the required estimate for $\beta=(0,0,0)$, the extension to arbitrary β being obvious. Given $|\alpha|=2$, and $|x|>2$, we can find $\phi_x\in C^\infty(\mathbb{R}^3)$ with $\mathrm{Supp}(\phi_x)\subset \overline{B(0,3)}\setminus B(0,1)$ such that:

$$\int_0^1 (1-t)y^\alpha D^\alpha \mathcal{U}^i(x+ty)dt =: \phi_x(y) \quad \forall\, y\in \mathrm{Supp}(\chi)$$

Asymptotic properties of $\mathcal{U}^i$ entail then that $\|\phi_x\|_{W^{1,\infty}(\mathbb{R}^3)}\leq C/|x|^3$. Therefore, we have, thanks to the uniform bound (16) and the embedding $\dot{H}^1(\mathbb{R}^3)\subset L^2_{loc}(\mathbb{R}^3)$:

$$\begin{aligned}\Big|\int_{\mathbb{R}^3} f_\chi[A,\mathcal{B}](y)\cdot\int_0^1(1-t)y^\alpha D^\alpha\mathcal{U}^i(x+ty)dtdy\Big| \\ \leq C\|f_\chi[A,\mathcal{B}]\|_{\dot{H}^{-1}(B(0,3)\setminus\overline{B(0,1)})}\|\phi_x\|_{H^1_0(B(0,3)\setminus\overline{B(0,1)})} \\ \leq C\|\bar{U}[A,\mathcal{B}]\|_{\dot{H}^1(B(0,3)\setminus\overline{B(0,1)})}\tfrac{1}{|x|^3}\leq C\|U[A,\mathcal{B}]\|_{\dot{H}^1(\mathbb{R}^3)}\tfrac{1}{|x|^3}\leq C\tfrac{|A|}{|x|^3}.\end{aligned}$$

This ends the proof of the proposition. □

We end this section by having a look to the interactions between the decomposition in Proposition 2.2 with scaling properties of the Stokes problem (14)-(15). Indeed, for any $a<1$, standard scaling arguments imply that:

$$U[A,a\mathcal{B}](x)=aU[A,\mathcal{B}](x/a), \qquad P[A,a\mathcal{B}](x)=P[A,\mathcal{B}](x/a).$$

Consequently for arbitrary $w\in\mathbb{S}^2$, we have:

$$\lim_{t\to\infty} t^2U[A,a\mathcal{B}](tw)=a^3\lim_{t\to\infty}t^2U[A,\mathcal{B}](tw)$$

This entails that $\mathbb{M}[A,a\mathcal{B}]=a^3\mathbb{M}[A,\mathcal{B}]$ and

$$\mathcal{K}[A,a\mathcal{B}](x)=a^3\mathcal{K}[A,\mathcal{B}](x). \tag{21}$$

We can then compare remainder terms. This yields:

$$|\nabla^\beta\mathcal{H}[A,a\mathcal{B}](x)|\leq \frac{C|A|a^4}{|x|^{3+|\beta|}} \quad \forall\,|x|>4a. \tag{22}$$

3. Approximation of the solution to the N-particle problem

In this section, we fix N large and $A \in \mathrm{Sym}_{3,\sigma}(\mathbb{R})$. We provide an approximation *via* the method of reflections for the solution (u_N, p_N) to

$$\begin{cases} -\mathrm{div}\, \Sigma_\mu(u,p) = 0 \\ \qquad \mathrm{div}\, u = 0 \end{cases} \qquad \text{in } \mathbb{R}^3 \setminus \bigcup_{l=1}^N \overline{B}_l, \tag{23}$$

$$\begin{cases} u(x) = V_l + \omega_l \times (x - x_l) & \text{on } \partial B_l, \text{ for } l = 1, \dots, N \\ u(x) = Ax & \text{at infinity,} \end{cases} \tag{24}$$

$$\begin{cases} \displaystyle\int_{\partial B_l} \Sigma_\mu(u,p) n \mathrm{d}s = 0 \\ \displaystyle\int_{\partial B_l} (x - x_l) \times \Sigma_\mu(u,p) n \mathrm{d}s = 0 \end{cases} \qquad \text{for } \; l = 1, \dots, N. \tag{25}$$

We recall that the method of reflections consists in matching the boundary conditions on each particle by solving a Stokes system around each particle, gluing together the local solutions into one approximation and iterating the process, since by gluing the local solutions we alter the boundary values of the approximation. More precisely, we first define

$$u_{app}^{(0)}(x) := Ax, \qquad A_l^{(0)} := A \text{ for } l = 1, \dots, N.$$

Given $n \geq 0$ and assuming that a vector-field $u_{app}^{(n)}$ and matrices $(A_l^{(n)})_{l=1,\dots,N}$ are constructed we set:

$$A_l^{(n+1)} := \sum_{l \neq \lambda} D(\mathcal{K})[A_\lambda^{(n)}, B_\lambda](x_l - x_\lambda) \qquad \forall\, l = 1, \dots, N \tag{26}$$

$$v^{(n+1)}(x) := \sum_{l=1}^N U[A_l^{(n)}, B_l](x - x_l) \qquad \text{on } \mathcal{F}^N \tag{27}$$

$$u_{app}^{(n+1)}(x) := u_{app}^{(n)}(x) + v^{(n+1)}(x) \qquad \text{on } \mathcal{F}^N. \tag{28}$$

Correspondingly, we compute a sequence of approximate pressure:

$$p_{app}^{(0)} = 0, \qquad p_{app}^{(n+1)} = p_{app}^{(n)} + \mu \sum_{l=1}^N P[A_l^{(n)}, B_l]\,, \qquad \forall\, n \in \mathbb{N}.$$

The factor μ is introduced here since (U, P) solves a Stokes system without viscosity.

The motivation of these definitions is the following remark. For each $n \geq 0$ the flow $v^{(n+1)}$ cancels the first order symmetric term of the leading part of the boundary value of $u_{app}^{(n)}$ on each ∂B_l. For instance, for $n = 0$, we notice that on ∂B_l, it holds:

$$\begin{aligned} u^{(0)}(x) &= Ax = Ax_l + A(x - x_l), \\ v^{(1)}(x) &= -A(x - x_l) + \sum_{l \neq \lambda} U[A, B_l](x - x_\lambda), \end{aligned}$$

which implies that:

$$u^{(1)} = Ax_l + \sum_{l\neq\lambda} U[A, B_l](x - x_\lambda).$$

Since ε_0 is very small, meaning that $a \ll d$, by Proposition 2.2 and (21), we have that for each $\lambda \neq l$ and any $x \in \partial B_l$:

$$U[A, B_l](x - x_\lambda) \;\; = \;\; a^3\mathcal{K}[A, \mathcal{B}_\lambda](x - x_\lambda) + \mathcal{H}[A, B_\lambda](x - x_\lambda).$$

where $\mathcal{H}[A, B_\lambda](x - x_\lambda) << \mathcal{K}[A, B_\lambda](x - x_\lambda)$ since $|x - x_l| >> a$ on ∂B_l. On the other hand, by Taylor expansion, for any $x \in \partial B_l$ and any $\lambda \neq l$,

$$a^3\mathcal{K}[A, \mathcal{B}_\lambda](x - x_\lambda) = \text{constant} + \text{rotation} + a^3 D(\mathcal{K})[A, \mathcal{B}_\lambda](x_l - x_\lambda)(x - x_l) + O(|x - x_l|^2).$$

Hence in the reflection method, we aim at canceling the symmetric gradient $a^3 D\mathcal{K}[A, \mathcal{B}_\lambda](x_l - x_\lambda)(x - x_l)$. By a direct iteration, we obtain that, for any $x \in \partial B_l$ and $n \geq 2$, there holds:

$$\begin{aligned} u^{(n)}_{app}(x) =& \text{constant} + \text{rotation} \\ &+ a^3 \sum_{l\neq\lambda} \mathcal{K}[A^{(n-1)}_\lambda, \mathcal{B}_\lambda](x - x_\lambda) \\ &+ a^3 \sum_{j=0}^{n-2} \sum_{l\neq\lambda} \Big(\mathcal{K}[A^{(j)}_\lambda, \mathcal{B}_\lambda](x - x_\lambda) - \mathcal{K}[A^{(j)}_\lambda, \mathcal{B}_\lambda](x_l - x_\lambda) \\ &\quad - [\nabla \mathcal{K}[A^{(j)}_\lambda, \mathcal{B}_\lambda](x_l - x_\lambda)](x - x_l) \Big) \\ &+ \sum_{j=0}^{n-1} \sum_{l\neq\lambda} \mathcal{H}[A^{(j)}_\lambda, B_\lambda](x - x_\lambda) \end{aligned} \tag{29}$$

The purpose of this section is twofold. First, we show that the method of reflections converges. We quantify then how close the family of approximations $(u^{(n)}_{app})_{n\in\mathbb{N}}$ are to the velocity-field u_N solution to (23)-(24)-(25).

We start with the convergence of the method. Since the correctors are fixed with respect to the family of matrices $(A^{(n)}_l)_{l=1,\dots,N}$ this amounts to prove that this family of matrices defines a converging series (in n for arbitrary $l \in \{1, \dots, N\}$). This is the content of the following proposition which relies mostly on item (1) of Lemma A.2 in Appendix A:

Proposition 3.1. *There exists $\varepsilon_0 > 0$ sufficiently small such that, for $\varepsilon < \varepsilon_0$ and $1 < q < \infty$, there exists a constant $C(q, \varepsilon_0)$ depending on q and ε_0, but independent of N, such that*

$$\Big(\sum_{l=1}^N |A^{(n+1)}_l|^q\Big)^{1/q} \leq C(q, \varepsilon_0) \left(\frac{a}{d}\right)^{3-3/q} \Big(\sum_{l=1}^N |A^{(n)}_l|^q\Big)^{1/q} \quad \forall\, n \in \mathbb{N}.$$

Proof. Let $n \geq 0$. We first notice that, by Proposition 2.2, there exists symmetric matrices $\mathbb{M}_l^{(n)} := \mathbb{M}[A_l^{(n)}, \mathcal{B}_l]$ such that

$$\mathcal{K}[A_l^{(n)}, B_l]_i = a^3 \mathbb{M}_l^{(n)} : \nabla \mathcal{U}^i = a^3 \sum_{k=1}^{3} [\mathbb{M}_l^{(n)}]_{kl} \partial_k U^{i,l}, \quad i = 1,2,3.$$

We remark then that, for $i,j \in \{1,2,3\}$, U^{ij} is homogeneous in $\mathbb{R}^3 \setminus \{0\}$ with degree -1. Moreover U^{ij} satisfies that

$$\Delta U^{ij} = \partial_i q_j \text{ in } \mathbb{R}^3 \setminus \{0\},$$

where for each $j \in \{1,2,3\}$, q_j is harmonic in $\mathbb{R}^3 \setminus \{0\}$. We can apply then Lemma A.2 to the computation of the components of $A_l^{(n+1)}$ by choosing $V := U^{ij}$ and $Q := \partial_i q_j$ for each $i,j \in \{1,2,3\}$. This yields that, for ε_0 sufficiently small and arbitrary $q \in (1,\infty)$

$$\Big(\sum_{l=1}^{N} |A_l^{(n+1)}|^q \Big)^{1/q} \leq C(q,\varepsilon_0) \left(\frac{a}{d}\right)^{3-3/q} \left(\sum_{l=1}^{N} |\mathbb{M}_l^{(n)}|^q \right)^{1/q}.$$

However, by Proposition 2.2, there exists an absolute constant (independent of n,l and other parameters) such that $|\mathbb{M}_l^{(n)}| \leq C|A_l^{(n)}|$. This completes the proof of the proposition. □

We proceed with the analysis of the quality of the sequence of approximations $(u_{app}^{(n)})_{n\in\mathbb{N}}$.

Proposition 3.2. *Let ε_0 sufficiently small. There exists a constant $C_{app}(\varepsilon_0)$, such that for $n \geq 3$ and $\varepsilon < \varepsilon_0$, there holds*

$$\|u_N - u_{app}^{(n)}\|_{\dot{H}^1(\mathbb{R}^3)} \leq C_{app}(\varepsilon_0)|A|\big(\frac{a}{d}\big)^{11/2}.$$

Proof. By substracting the equations satisfied by (u_N, p_N) and $(u_{app}^{(n)}, p_{app}^{(n)})$, we obtain that $\delta_u = u_N - u_{app}^{(n)}$, $\delta_p = p_N - p_{app}^{(n)}$ satisfies:

$$\begin{cases} -\operatorname{div} \Sigma_\mu(\delta_u, \delta_p) = 0 \\ \operatorname{div} \delta_u = 0 \end{cases} \qquad \text{in } \mathbb{R}^3 \setminus \bigcup_{l=1}^{N} \overline{B}_l, \tag{30}$$

and

$$\begin{cases} \displaystyle\int_{\partial B_l} \Sigma_\mu(\delta_u, \delta_p) n \mathrm{d}s = 0 \\ \displaystyle\int_{\partial B_l} (x - x_l) \times \Sigma_\mu(\delta_u, \delta_p) n \mathrm{d}s = 0 \end{cases} \qquad \text{for } l = 1, \ldots, N. \tag{31}$$

As for boundary conditions, we note that by definition of $u_{app}^{(n)}$, we have that

$$\delta_u(x) = u_N(x) - Ax - \sum_{j=1}^{n} v^{(j)}(x), \qquad \text{in } \mathcal{F}_N$$

Thanks to (17) and extending the velocities $u_{app}^{(n)}$ and u_N inside the particle domains with their boundary values, we have that $\delta_u \in \dot{H}^1(\mathbb{R}^3)$. On the boundaries, reorganizing the terms involved in $v^{(j)}$, see also (29), we have that there exists vectors $(W_l, \varpi_l)_{l=1,\dots,N}$ for which:

$$\delta_u(x) = W_l + \varpi_l \times (x - x_l) + u^*_{l,n}, \text{ on } B_l, \ \forall l = 1, \dots, N, \tag{32}$$

where we have $u^*_{l,n} = S_l^{(n)} + R_l^{(n)}$ with:

$$S_l^{(n)}(x) := \sum_{l \neq \lambda} a^3 \mathcal{K}[A_\lambda^{(n-1)}, \mathcal{B}_\lambda](x - x_\lambda),$$

and

$$\begin{aligned} R_l^{(n)}(x) \quad := \quad & \sum_{j=0}^{n-2} \sum_{l \neq \lambda} a^3 \mathcal{K}[A_\lambda^{(j)}, \mathcal{B}_\lambda](x - x_\lambda) - \sum_{j=0}^{n-2} \sum_{l \neq \lambda} a^3 \mathcal{K}[A_\lambda^{(j)}, \mathcal{B}_\lambda](x_l - x_\lambda) \\ & - \sum_{j=0}^{n-2} \sum_{l \neq \lambda} a^3 [\nabla \mathcal{K}[A_\lambda^{(j)}, \mathcal{B}_\lambda](x_l - x_\lambda)](x - x_l) + \sum_{j=0}^{n-1} \sum_{l \neq \lambda} \mathcal{H}[A_\lambda^{(j)}, B_\lambda](x - x_\lambda). \end{aligned}$$

We notice that for each $l = 1, \dots, N$, the formula defining $u^*_{l,n}$ can be extended to $B(x_l, 2a)$. We also mention that again classical integration by parts arguments yield that δ_u realizes

$$\min \left\{ \int_{\mathcal{F}_N} |D(v)|^2, \ v \in \dot{H}^1(\mathbb{R}^3), \quad \text{div } v = 0, \ D(v - u^*_{l,n}) = 0 \text{ on } B_l, \ \forall \, l \right\}. \tag{33}$$

The proof of our theorem then reduces to construct divergence-free vector-fields $w_{l,n} \in C_c^\infty(B(x_l, 2a))$ that match $u^*_{l,n}$ (up to a rigid vector-field) on B_l for each $l = 1, \dots, N$. Indeed, since δ_u is divergence-free and using the minimizing principle of (33), we have then:

$$\int_{\mathbb{R}^3} |\nabla \delta_u|^2 = 2 \int_{\mathbb{R}^3} |D(\delta_u)|^2 \leq C \sum_{l=1}^N \int_{B(x_l, 2a)} |\nabla w_{l,n}|^2.$$

So, we define:

$$w_{l,n}(x) := \sum_{l=1}^N \big(\chi(\frac{x - x_l}{a})(u^*_{l,n}(x) - \bar{u}^*_{l,n}) - \mathfrak{B}_{B(x_l, 2a) \setminus \overline{B(x_l, a)}}[(u^*_{l,n}(x) - \bar{u}^*_{l,n})] \cdot \nabla \chi(\frac{x - x_l}{a}) \big),$$

Here, we denoted $\chi \in C_0^\infty(\mathbb{R}^3)$ such that $\chi \equiv 1$ on $B(0, 3/2)$ and $\chi \equiv 0$ in $\mathbb{R}^3 \setminus \overline{B(0, 2)}$, $\bar{u}^*_{l,n}$ is the mean-value of $u^*_{l,n}$ over $B(x_l, 2a)$. Clearly, our candidate matches the condition

$$w_{l,n}(x) = \text{constant} + u^*_{l,n}, \quad \text{on } B_l.$$

For the next computations, we introduce also $\bar{S}_l^{(n)}$ and $\bar{R}_l^{(n)}$ the mean-values of $S_l^{(n)}$ and $R_l^{(n)}$ over $B(x_l, 2a)$ respectively so that $\bar{u}^*_{l,n} = \bar{S}_l^{(n)} + \bar{R}_l^{(n)}$.

By the scaling properties of the Bogovskii operator, we obtain that

$$\begin{aligned}\int_{\mathcal{F}_N} |\nabla w_{l,n}|^2 &\leq C\sum_{l=1}^{N} \|\nabla(\chi(\frac{\cdot - x_l}{a})(u^*_{l,n} - \bar{u}^*_{l,n}))\|^2_{L^2(B(x_l,2a))}\\ &\lesssim \sum_{j=1}^{4} H^{(n)}_{l,j},\end{aligned}$$

where

$$H^{(n)}_{l,1} := \sum_{l=1}^{N} \|\nabla S^{(n)}_l\|^2_{L^2(B(x_l,2a))},\ H^{(n)}_{l,2} := \sum_{l=1}^{N} \|\nabla R^{(n)}_l\|^2_{L^2(B(x_l,2a))},$$

and

$$H^{(n)}_{l,3} := \frac{1}{a^2}\sum_{l=1}^{N} \|S^{(n)}_l - \bar{S}^{(n)}_l\|^2_{L^2(B(x_l,2a))},\ H^{(n)}_{l,4} := \frac{1}{a^2}\sum_{l=1}^{N} \|R^{(n)}_l - \bar{R}^{(n)}_l\|^2_{L^2(B(x_l,2a))}.$$

Here, it is standard that the Poincaré-Wirtinger inequality entails that $H^{(n)}_{l,3} \leq CH^{(n)}_{l,1}$ and $H^{(n)}_{l,4} \leq CH^{(n)}_{l,2}$. Hence, we only need to bound $H^{(n)}_{l,1}$ and $H^{(n)}_{l,2}$.

We deal with $H^{(n)}_{l,1}$ first. According to the definition of $S^{(n)}_l$ and $\mathcal{K}[A^{(n-1)}_\lambda, \mathcal{B}_\lambda]$ (see Proposition 2.2), for each $l = 1, \dots N$, we have

$$\begin{aligned}S^{(n)}_l &= \sum_{l\neq\lambda} a^3\mathcal{K}[A^{(n-1)}_\lambda, \mathcal{B}_\lambda](x - x_\lambda)\\ &= a^3\sum_{l\neq\lambda}(\mathbb{M}[A^{(n-1)}_\lambda, \mathcal{B}_\lambda] : \nabla\mathcal{U}^1(x), \mathbb{M}[A^{(n-1)}_\lambda, \mathcal{B}_\lambda] : \nabla\mathcal{U}^2(x), \mathbb{M}[A^{(n-1)}_\lambda, \mathcal{B}_\lambda] : \nabla\mathcal{U}^3(x)).\end{aligned}$$

As in the proof of Proposition 3.1, for each $i, j \in \{1, 2, 3\}$, U^{ij} is homogeneous in $\mathbb{R}^3 \setminus \{0\}$ with degree -1 such that

$$\Delta U^{ij} = \partial_i q_j \text{ in } \mathbb{R}^3 \setminus \{0\},$$

where for each $j \in \{1, 2, 3\}$, q_j is harmonic in $\mathbb{R}^3 \setminus \{0\}$. By the definition of $A^{(n+1)}_l$ and applying Lemma A.2 by choosing $V := U^{ij}$ and $Q := \partial_i q_j$ for each $i, j \in \{1, 2, 3\}$ and Proposition 3.1, we have

$$H^{(n)}_{l,1} \leq [C(2, \varepsilon_0)]^{n-1}\big(\frac{a}{d}\big)^{3n}(a^3N)|A|^2.$$

Up to restrict the size of ε_0 we obtain that:

$$H^{(n)}_{l,1} \leq C(\varepsilon_0)\big(\frac{a}{d}\big)^{8}(a^3N)|A|^2. \tag{34}$$

Now we turn to deal with $H_{l,2}^{(n)}$. By the definition of $R_l^{(n)}$, we have that for any $x \in B(x_l, 2a)$,

$$|\nabla R_l^{(n)}(x)| \leq Ca^4 \sum_{j=0}^{n-1} \sum_{l \neq \lambda} |A_\lambda^{(j)}| \frac{1}{|x_l - x_\lambda|^4}.$$

We notice here – since the minimum distance between two x_l's is lager than d which is much larger than a (for small ε_0) – that, for each $l = 1, \ldots, N$ and for any $x \in B(x_l, 2a)$, there holds:

$$\begin{aligned} \sum_{l \neq \lambda} |A_\lambda^{(j)}| \frac{1}{|x_l - x_\lambda|^4} &\leq Cd^{-3} \sum_{l \neq \lambda} \int_{B(x_\lambda, d/2)} \frac{|A_\lambda^{(j)}|}{|x_l - y|^4} dy \\ &\leq Cd^{-3} \int_{\mathbb{R}^3} |\Phi^{(j)}(y)| \frac{\mathbf{1}_{|x_l - y| > d/2}}{|x_l - y|^4} dy \\ &\leq C(\varepsilon_0) d^{-3} \int_{\mathbb{R}^3} |\Phi^{(j)}(y)| \frac{\mathbf{1}_{|x - y| > d/2}}{|x - y|^4} dy, \end{aligned}$$

where

$$\Phi^{(j)}(x) := \sum_{l=1}^N A_l^{(j)} \mathbf{1}_{B(x_l, d/2)}(x).$$

Therefore we obtain, with a direct Young inequality for convolution:

$$H_{l,2}^{(n)} \leq C \frac{a^8}{d^6} \| \sum_{j=0}^{n-1} |\Phi^{(j)}| * \frac{\mathbf{1}_{|y| > d/2}}{|y|^4} \|_{L^2(\mathbb{R}^3)}^2 \leq C \frac{a^8}{d^8} \Big(\sum_{j=0}^{n-1} \|\Phi^{(j)}\|_{L^2(\mathbb{R}^3)} \Big)^2,$$

which combined with Proposition 3.1, yields that:

$$H_{l,2}^{(n)} \leq \left(\frac{a}{d}\right)^8 \sum_{j=0}^{n-1} \left(C(2, \varepsilon_0) \left(\frac{a}{d}\right)^{3/2} \right)^{2j} Na^3 |A|^2 \leq C(\varepsilon_0) \Big(\frac{a}{d}\Big)^8 Na^3 |A|^2, \tag{35}$$

where we have chosen ε_0 sufficiently small so that the series $(\sum_{j \geq 0} C(2, \varepsilon_0)(a/d)^{3/2})^{2j}$ converges. Combining (34)-(35), we obtain the expected result. □

4. Approximation of the target system

In this section, we fix $A \in \mathrm{Sym}_{3,\sigma}(\mathbb{R})$ and $\mathbb{M}_{eff} \in \mathcal{M}(\varepsilon_0)$ (for some small ε_0) and we analyse the properties of the asymptotic problem

$$\left\{ \begin{aligned} -\mathrm{div}(2\mu(1 + \mathbb{M}_{eff})(D(u)) - p\mathbb{I}_3) &= 0 \\ \mathrm{div} u &= 0 \end{aligned} \right. \quad \text{on } \mathbb{R}^3, \tag{36}$$

$$u(x) = Ax \quad \text{at infinity.} \tag{37}$$

We note that μ is a gain a simple factor in this equation so that we only treat the case $\mu = 1$ below. This system is associated with the weak formulation:

Find $v \in \dot{H}^1_\sigma(\mathbb{R}^3)$ such that:

$$2\mu \int_{\mathbb{R}^3} [(1+\mathbb{M}_{eff})(D(v)+A)] : \nabla w = 0\,, \quad \forall\, w \in \dot{H}^1_\sigma(\mathbb{R}^3).$$

Since $\mathbb{M}_{eff}$ has compact support, for ε_0 sufficiently small we have that $\|\mathbb{M}_{eff}\|_{L^\infty(\mathbb{R}^3)} \leq 1/2$ so that construction of a weak solution falls into the scope of the Lax-Milgram theorem. Hence, under the assumption that ε_0 is sufficiently small we have existence and uniqueness of a u_c satisfying

- $v_c(x) = (u_c(x) - Ax) \in \dot{H}^1_\sigma(\mathbb{R}^3)$,
- there exists p_c for which (36) holds true (in $\mathcal{D}'(\mathbb{R}^3)$),

and consequently, the pressure p_c exists and is unique up to a constant. We focus now – as in the previous section for the problem in a perforated domain – on a possible expansion of the solution u_c in terms of "powers of $\mathbb{M}_{eff}$". Namely, for small ε_0 the matrix $\mathbb{M}_{eff}$ can be seen as a perturbation of the identity so that one may look for a solution to (36)-(37) by iterating the mapping $v \mapsto \tilde{v} := \mathcal{L}v$ solving the system:

$$\left\{\begin{aligned} -\mathrm{div}(D(\tilde{v}) - p\mathbb{I}_3) &= \mathrm{div}\,\mathbb{M}_{eff}(D(v)) + \mathrm{div}\,\mathbb{M}_{eff}(A) \\ \mathrm{div}\tilde{v} &= 0 \end{aligned}\right. \qquad \text{in } \mathbb{R}^3,$$
$$\tilde{v}(x) = 0 \qquad \text{at infinity,}$$

starting form $v^{(0)}(x) = 0$. Again, it is standard by introducing a weak formulation and Lax-Milgram arguments that there exists a unique velocity-field $\tilde{v}_c$ satisfying

- $\tilde{v}_c \in \dot{H}^1_\sigma(\mathbb{R}^3)$
- there exists a pressure $\tilde{p}_c$ such that

$$\left\{\begin{aligned} -\mathrm{div}(D(\tilde{v}_c) - \tilde{p}_c\mathbb{I}_3) &= \mathrm{div}\,\mathbb{M}_{eff}(A) \\ \mathrm{div}\tilde{v} &= 0 \end{aligned}\right. \qquad \text{in } \mathbb{R}^3,$$

The main result of this section is the following proposition which compares the velocity-field $v_c(x) = u_c(x) - Ax$ with $\tilde{v}_c$.

Proposition 4.1. *Under the assumption that $\varepsilon_0 > 0$ is sufficiently small, there exists a constant $C(K,\varepsilon_0)$ such that*

$$\|\nabla v_c - \nabla \tilde{v}_c\|_{L^2(\mathbb{R}^3)} \leq C(K,\varepsilon_0)\|\mathbb{M}_{eff}\|^2_{L^\infty(\mathbb{R}^3)}|A|. \tag{38}$$

Proof. This proof is a straightforward application of fixed-point arguments. First, let prove that the mapping $\mathcal{L}$ is a contraction on $\dot{H}^1_\sigma(\mathbb{R})$. Indeed, for arbitrary $(v_1, v_2) \in \dot{H}^1_\sigma(\mathbb{R}^3)$, given the weak formulation for the Stokes problem, we have that $w = \mathcal{L}(v_1 - v_2)$ satisfies:

$$\int_{\mathbb{R}^3} D(w) : D(\varphi) = \int_{\mathbb{R}^3} [\mathbb{M}_{eff}(D(v_1 - v_2))] : D(\varphi)\,, \qquad \forall\, \varphi \in \dot{H}^1_\sigma(\mathbb{R}^3).$$

Setting $w = \varphi$ and recalling that w is divergence-free we obtain that

$$\int_{\mathbb{R}^3} |\nabla w|^2 = 2\int_{\mathbb{R}^3} |D(w)|^2 \leq 8\|\mathbb{M}_{eff}\|^2_{L^\infty} \int_{\mathbb{R}^2} |D(v_1 - v_2)|^2 \leq 4\|\mathbb{M}_{eff}\|^2_{L^\infty} \int_{\mathbb{R}^2} |\nabla v_1 - \nabla v_2|^2.$$

Consequently, fix $\varepsilon_0 < 1/8$. Then $\|\mathcal{L}\| < 1/\sqrt{2}$ and the mapping $\mathcal{L}$ is a contraction that admits a unique fixed point. This yields a solution to (36)-(37). Furthermore, this solution is obtained by iterating the mapping $\mathcal{L}$ from $v^{(0)} = 0$. So the sequence $v^{(n)} = \mathcal{L}^{\circ n} v^{(0)}$ converges to v_c (in $\dot{H}^1(\mathbb{R}^3)$) while, by definition, $\tilde{v}_c = v^{(1)}$. Similar energy estimates yield that

$$\|\nabla \tilde{v}_c\|_{L^2(\mathbb{R}^3)} \leq |K|^{1/2} \|M_{eff}\|_{L^\infty(\mathbb{R}^3)} |A|. \tag{39}$$

Standard arguments with contractions then yield that

$$\begin{aligned}\|\nabla \tilde{v}_c - \nabla v_c\|_{L^2(\mathbb{R}^3)} &= \|v^{(1)} - \lim v^{(n)}\|_{\dot{H}^1(\mathbb{R}^3)} \\ &\leq \frac{\|\mathcal{L}\|}{1 - \|\mathcal{L}\|} \|v^{(1)}\|_{\dot{H}^1(\mathbb{R}^3)} \leq 4|K|^{1/2} \|\mathbb{M}_{eff}\|_{L^\infty(\mathbb{R}^3)} \|M_{eff}\|_{L^\infty(\mathbb{R}^3)} |A|.\end{aligned}$$

This concludes the proof. □

5. Proof of main result

We end the paper with a proof of Theorem 1.1. In the whole section, we assume that we are given a perforated domain such that (H1)-(H2) hold true. We are also given $\mathbb{M}_{eff} \in \mathcal{M}(\varepsilon_0)$ with simultaneously $(a/d)^3 < \varepsilon_0$ (see (H1)-(H2) for the definitions of a and d). Restrictions on ε_0 are introduced throughout the section. Finally, we fix a matrix $A \in \mathrm{Sym}_{3,\sigma}(\mathbb{R})$.

We recall that Theorem 1.1 is a stability estimate between the solutions to two problems. The first one is the Stokes problem in a perforated domain (2)-(3)-(4) that we studied in Section 3. We restrict at first ε_0 so that Proposition 3.2 holds true. We have then a sequence of approximations $(u_{app}^{(n)})_{n\in\mathbb{N}}$ to the velocity-field $u_N(x) = Ax + v_N(x)$ solution to (2)-(3)-(4). The second problem is the continuous analogue (5)-(6) that we studied in Section 4. We assume also that ε_0 is sufficiently small so that Proposition 4.1 holds true. We have then an approximation $\tilde{u}_c(x) = Ax + \tilde{v}_c(x)$ to the velocity-field u_c solution to (5)-(6).

The purpose of Theorem 1.1 is to compute a bound from above for $u_N - u_c$. To this end, we fix $n = 3$ and $u_{app} = u_{app}^{(3)}$ (with the notations of Proposition 3.2) and write

$$u_N - u_c = (u_N - u_{app}) + (u_{app} - \tilde{u}_c) + (\tilde{u}_c - u_c) =: R_{perf} + R_{main} + R_{cont}. \tag{40}$$

The two error terms R_{perf} and R_{cont} have been estimated previously in Proposition 3.2 and Proposition 4.1 respectively. So, we proceed in the next subsection with estimating R_{main} and shall combine the various partial results in the last subsection to complete the proof of our theorem.

5.1. **Computing R_{main}.** We recall that $u_{app} = u_{app}^{(3)}$ is constructed via the method of reflections:

$$u_{app}(x) = Ax + \sum_{j=1}^{3} \Big(\sum_{l=1}^{N} U[A_l^{(j-1)}, B_l](x - x_l) \Big).$$

By the definition of u_{app} and Proposition 2.2, we have the following decomposition, for any $x \in \mathcal{F}_N$

$$u_{app}(x) - \tilde{u}_c(x) = R_1(x) + R_2(x), \tag{41}$$

where

$$R_1(x) := \sum_{l=1}^{N} \mathcal{K}[A, B_l](x - x_l) - \tilde{v}_c(x)$$

and

$$R_2(x) := \sum_{l=1}^{N} \mathcal{H}[A, B_l](x - x_l) + \sum_{j=2,3} \sum_{l=1}^{N} U[A_l^{(j-1)}, B_l](x - x_l).$$

We start with computing R_1 :

Proposition 5.1. *Let $K_0 \Subset \mathbb{R}^3$ and $p \in [1, 3/2[$, there exists $C(K_0)$ for which:*

$$\|R_1\|_{L^p(K_0 \setminus \bigcup B(x_l, 4a))} \leq C(K_0) \Big[\|\mathbb{M}_N - \mathbb{M}_{eff}\|_{\dot{H}^{-1}(\mathbb{R}^3)} + \left(\frac{a^3}{d^3} \right)^{1+\theta} \Big], \tag{42}$$

where $\theta = \frac{1}{p} - \frac{2}{3}$.

Proof. By Proposition 2.2, we know that each component of $\mathcal{K}$ can be written as:

$$\mathcal{K}[A, B_l]_i(x - x_l) \quad = \quad \mathbb{M}[A, B_l] : \nabla \mathcal{U}^i(x - x_l)$$

In this identity, we recall that $\mathbb{M}[A, B_l] = a^3 \mathbb{M}[A, \mathcal{B}_l]$, given by (19), and that $\mathcal{U}^i = (U^{i1}, U^{i2}, U^{i3})$ with

$$U^{ij} := -\frac{1}{8\pi} \Big[\frac{\delta_{ij}}{|x - y|} + \frac{(x_i - y_i)(x_j - y_j)}{|x - y|^3} \Big]$$

corresponding to the fundamental solution of Stokes equation in $\mathbb{R}^3$. According to the fact that for any $x \in \mathbb{R}^3 \setminus \{0\}$

$$\Delta \mathcal{U}^i(x) = \nabla q_i$$

with $q_i(z) = \frac{1}{4\pi} \frac{z}{|z|^3}$ and applying Lemma A.1, we obtain that for any $x \in K_0 \setminus \bigcup B(x_\lambda, a)$ and $i = 1, 2, 3$, for any $l = 1, \dots, N$:

$$\begin{aligned} \mathcal{K}[A, B_l]_i(x - x_l) = \frac{3}{4\pi} \mathbb{M}[A, \mathcal{B}_l] : \int_{B(x_l, a)} \nabla \mathcal{U}^i(x - y) dy \\ + \frac{a^3}{3} \mathbb{M}[A, \mathcal{B}_l] : \int_0^a \left(\frac{r^4}{a^3} - r \right) ⨍_{B(x_l, r)} \nabla^2 q_i(x - y) dy dr \end{aligned}$$

We proceed by remarking that:

$$\sum_{l=1}^{N} \frac{3}{4\pi} \mathbb{M}[A, \mathcal{B}_l] : \int_{B(x_l, a)} \nabla \mathcal{U}^i(x - y) dy = \int_{\mathbb{R}^3} \mathbb{M}_N(A)(y) : \nabla \mathcal{U}^i(x - y) dy \tag{43}$$

Furthermore, since q_i is harmonic on $\mathbb{R}^3 \setminus \{0\}$, for $x \in K_0 \setminus \bigcup B(x_\lambda, a)$ and $l \in \{1, \dots, N\}$ there holds:

$$
\begin{aligned}
\int_0^a \left(\frac{r^4}{a^3} - r\right) \fint_{B(x_l,r)} \nabla^2 q_i(x-y)dydr &= \int_0^a \left(\frac{r^4}{a^3} - r\right) \fint_{B(x_l,a)} \nabla^2 q_i(x-y)dydr \\
&= -\frac{9}{40\pi a} \int_{B(x_l,a)} \nabla^2 q_i(x-y)dy.
\end{aligned} \tag{44}
$$

On the other hand, by uniqueness of the solution to the Stokes problem defining $\tilde{v}_c$ (in $\dot{H}^1(\mathbb{R}^3)$), we know that $\tilde{v}_c$ is computed with Green's function for the Stokes problem. This yield componentwise:

$$
\tilde{v}_{c,i}(x) = \int_{\mathbb{R}^3} \mathbb{M}_{eff}(A)(y) : \nabla \mathcal{U}^i(x-y)\mathrm{d}y, \quad \forall\, x \in \mathbb{R}^3.
$$

We note that this quantity is well-defined since $\mathbb{M}_{eff} \in L^\infty(\mathbb{R}^3)$ has compact support and $\nabla \mathcal{U}^i$ is homogeneous of degree -2 so that it is $L^p_{loc}(\mathbb{R}^3)$ for arbitrary $p < 3/2$. Eventually, the $i-$th component of R_1 can be rewritten as

$$
\begin{aligned}
R_{1,i}(x) = \int_{\mathbb{R}^3} \left[\mathbb{M}_N - \mathbb{M}_{eff}\right](A)(y) : \nabla \mathcal{U}^i(x-y)dy \\
- \frac{3a^2}{40\pi} \int_{\mathbb{R}^3} \mathbb{M}_N(A)(y) : \mathbf{1}_{|x-y|>3a} \nabla^2 q_i(x-y)dy,
\end{aligned}
$$

since $x \notin \bigcup B(x_l, 4a)$. Concerning the first term on the right-hand side of this equality, let denote h any component of $\left[\mathbb{M}_N - \mathbb{M}_{eff}\right]$. By assumption, we have then $h \in \dot{H}^{-1}(\mathbb{R}^3)$ so that it can be written $h = \partial_1 \varphi_1 + \partial_2 \varphi_2$, where φ_1 and φ are $L^2(\mathbb{R}^3)$ and

$$
\max_{i=1,2} \|\varphi_i\|_{L^2(\mathbb{R}^3)} \leq \|h\|_{\dot{H}^{-1}(\mathbb{R}^3)} \leq \|\mathbb{M}_N - \mathbb{M}_{eff}\|_{\dot{H}^{-1}(\mathbb{R}^3)}.
$$

Therefore, we have on $\mathbb{R}^3$:

$$
\int_{\mathbb{R}^3} h(y)\partial_l U^{ik}(x-y)dy = \int_{\mathbb{R}^3} \varphi_1(y)\partial_{1l} U^{ik}(x-y)dy + \int_{\mathbb{R}^3} \varphi_2(y)\partial_{2l} U^{ik}(x-y)dy,
$$

where, by Calderón Zygmund inequality, the right-hand side of this identity is well defined and satisfies.

$$
\| \int_{\mathbb{R}^3} h \partial_l U^{ik}(x-y)dy \|_{L^2(\mathbb{R}^3)} \leq C \|\mathbb{M}_N - \mathbb{M}_{eff}\|_{\dot{H}^{-1}(\mathbb{R}^3)}. \tag{45}
$$

As for the second term in $R_{1,i}$ we can apply a classical Young inequality for convolution to obtain:

$$
\begin{aligned}
\| \int_{\mathbb{R}^3} \mathbb{M}_N(A)(y) \nabla^2 q_i(x-y)dy \|_{L^p(\mathbb{R}^3)} &\leq C \|\mathbb{M}_N(A)\|_{L^p(\mathbb{R}^3)} \| \frac{\mathbf{1}_{|z|>3a}}{|z|^4} \|_{L^1(\mathbb{R}^3)} \\
&\leq C|A| \Big(\frac{a}{d}\Big)^{\frac{3}{p}} \frac{1}{a}.
\end{aligned}
$$

Finally, applying the embedding $L^2(K_0) \subset L^p(K_0)$, we have:

$$\|R_1\|_{L^p(K_0 \subset \bigcup B(x_l,4a))} \leq C|A| \left[\|\mathbb{M}_N(y) - \mathbb{M}_{eff}(y)\|_{\dot{H}^{-1}(\mathbb{R}^3)} + a\Big(\frac{a}{d}\Big)^{\frac{3}{p}}. \right]$$

We conclude by applying again that $Nd^3 \leq |K|$ so that

$$a\left(\frac{a}{d}\right)^{\frac{3}{p}} \leq (Na^3)^{\frac{1}{3}} \left(\frac{a}{d}\right)^{\frac{3}{p}} \leq \left[\left(\frac{a}{d}\right)^3\right]^{\frac{1}{p}+\frac{1}{3}}.$$

□

We proceed by computing error estimates for R_2. This is the content of the following proposition:

Proposition 5.2. *Let $K_0 \Subset \mathbb{R}^3$ and $p \in [1, 3/2[$, there exists $C(K, K_0)$ for which:*

$$\|R_2\|_{L^p(K_0 \setminus \bigcup B(x_l,4a))} \leq C(K, K_0) \left(\frac{a^3}{d^3}\right)^{1+\theta}, \tag{46}$$

where $\theta = \frac{1}{p} - \frac{2}{3}$.

Proof. We recall that

$$\begin{aligned} R_2(x) &= \sum_{l=1}^N \mathcal{H}[A, B_l](x - x_l) + \sum_{j=2,3} \sum_{l=1}^N U[A_l^{(j-1)}, B_l](x - x_l) \\ &= \sum_{j=2,3} \sum_{l=1}^N \mathcal{K}[A_l^{(j-1)}, B_l](x - x_l) + \sum_{j=1}^3 \sum_{l=1}^N \mathcal{H}[A_l^{(j-1)}, B_l](x - x_l). \end{aligned}$$

Since $B_l = a\mathcal{B}_l$, by Proposition 2.2, we know that for any $j = 1, 2, 3$ and $l = 1, \dots, N$, when $|x - x_l| > 4a$:

$$|\mathcal{K}[A_l^{(j-1)}, B_l](x - x_l)| \leq C \frac{a^3 |A_l^{(j)}|}{|x - x_l|^2}, \qquad |\mathcal{H}[A_l^{(j-1)}, B_l](x - x_l)| \leq C \frac{a^4 |A_l^{(j)}|}{|x - x_l|^3}.$$

Therefore, on $K_0 \setminus \bigcup B(x_l, 4a)$, we have:

$$|R_2(x)| \leq C \Big[a^3 \sum_{l=1}^N \frac{|A_l^{(1)}| + |A_l^{(2)}|}{|x - x_l|^2} + a^4 \sum_{l=1}^N \frac{|A|}{|x - x_l|^3} \Big].$$

Consequently, denoting $K_1 = K_0 \cup K$, we obtain

$$\begin{aligned}\|R_2\|_{L^p(K_0)} &\leq Ca^3 \sum_{i=1}^{2}\sum_{l=1}^{N} |A_l^{(i)}| \left(\int_a^{diam(K_1)} \frac{dx}{|x|^{2p}}\right)^{\frac{1}{p}} + Na^4|A| \left(\int_a^{\infty} \frac{dx}{|x|^{3p}}\right)^{\frac{1}{p}} \\ &\leq C(K_1)\left(a^{1+\frac{3}{p}} \sum_{i=1}^{2}\sum_{l=1}^{N} |A_l^{(i)}| + Na^{1+\frac{3}{p}}|A|\right) \\ &\leq C(K,K_1)\left(\frac{a^3}{d^3}\right)^{1+\theta} |A|\end{aligned}$$

where we applied Proposition 3.2 to pass from the second to the last line together with the remark that $Nd^3 \leq |K|$. □

5.2. **End of the proof.** Let $K_0 \Subset \mathbb{R}^3$ containing K (for simplicity) and $p \in [1, 3/2[$. By (40) we have

$$\|u_N - u_c\|_{L^p(K_0)} \leq \|R_{perf}\|_{L^p(K_0)} + \|R_{main}\|_{L^p(K_0)} + \|R_{cont}\|_{L^p(K_0)}.$$

Since $p \leq 6$ and by the embedding $\dot{H}^1(\mathbb{R}^3) \subset L^p_{loc}(\mathbb{R}^3)$ and Proposition 3.2, we have the bounds

$$\begin{aligned}\|R_{perf}\|_{L^p(K_0)} &\leq C(K_0)\|R_{perf}\|_{L^6(\mathbb{R}^3)} \leq C(K_0)\|u_N - u_{app}\|_{\dot{H}^1(\mathbb{R}^3)} \\ &\leq C(\varepsilon_0, K_0)|A| \left(\frac{a^3}{d^3}\right)^{\frac{11}{6}}.\end{aligned}$$

With a similar chain of inequalities, we obtain by applying Proposition 4.1

$$\|R_{cont}\|_{L^p(K_0)} \leq C(K, K_0, \varepsilon_0)|A| \|\mathbb{M}_{eff}\|^2_{L^\infty(\mathbb{R}^2)}.$$

Finally, concerning R_{main}, we recall that we have simultaneously $R_{main} = u_{app} - \tilde{u}_c = v_{app} - \tilde{v}_c$ (where we denote $v_{app}(x) = u_{app}(x) - Ax$) and $R_{main} = R_1 + R_2$, with the notations of the previous subsection. This entails that:

$$\|R_{main}\|_{L^p(K_0)} \leq \|u_{app} - \tilde{u}_c\|_{L^p(\bigcup B(x_l,4a))} + \|R_1\|_{L^p(K_0 \setminus \bigcup B(x_l,4a))} + \|R_2\|_{L^p(K_0 \setminus \bigcup B(x_l,4a))}.$$

The two last terms of the right-hand side are controlled respectively by (42) and (46):

$$\begin{aligned}(47)\quad &\|R_1\|_{L^p(K_0 \setminus \bigcup B(x_l,4a))} + \|R_2\|_{L^p(K_0 \setminus \bigcup B(x_l,4a))} \\ &\qquad\leq C(K,K_0)|A| \left[\|\mathbb{M}_N - \mathbb{M}_{eff}\|_{\dot{H}^{-1}(\mathbb{R}^3)} + \left(\frac{a^3}{d^3}\right)^{1+\theta}\right]\end{aligned}$$

where $\theta = \frac{1}{p} - \frac{2}{3}$. As for the first term, we first bound

$$\|u_{app} - \tilde{u}_c\|_{L^p(\cup B(x_l,4a))} \leq |\bigcup B(x_l, 4a)|^{\frac{1}{p}-\frac{1}{6}} (\|v_{app}\|_{L^6(\mathbb{R}^3)} + \|\tilde{v}_c\|_{L^6(\mathbb{R}^3)}).$$

Here, it is straightforward from (39) that:

$$\|\tilde{v}_c\|_{L^6(\mathbb{R}^3)} \leq C(K)|A| \|\mathbb{M}_{eff}\|_{L^\infty(\mathbb{R}^3)}.$$

As for u_{app}, we have, by Proposition 3.2 and uniform estimate (18) that:

$$\begin{aligned}\|v_{app}\|_{L^6(\mathbb{R}^3)} &\le C\|\nabla v_{app}\|_{L^6(\mathbb{R}^3)} \le C\left(\|\nabla v_{app} - \nabla v_N\|_{L^2(\mathbb{R}^3)} + \|\nabla v_N\|_{L^2(\mathbb{R}^3)}\right)\\ &\le C|A|\left(\frac{a^3}{d^3}\right)^{\frac{1}{2}}.\end{aligned}$$

Via a straightforward bound on the volume of the $B(x_l,4a)$ we conclude that:

$$\|u_{app} - \tilde{u}_c\|_{L^p(\bigcup B(x_l,4a))} \le \left(\frac{a^3}{d^3}\right)^{\frac{1}{p}-\frac{1}{6}}\left(\left(\frac{a^3}{d^3}\right)^{1/2} + \|\mathbb{M}_{eff}\|_{L^\infty(\mathbb{R}^3)}\right)|A|. \tag{48}$$

Combining (47) and (48) yields

$$\|R_{main}\|_{L^p(K_0)} \le C(K,K_0,\varepsilon_0)|A|\left[\|\mathbb{M}_N - \mathbb{M}_{eff}\|_{\dot{H}^{-1}(\mathbb{R}^3)} + \left(\frac{a^3}{d^3}\right)^{1+\theta} + \|\mathbb{M}_{eff}\|^2_{L^\infty(\mathbb{R}^3)}\right], \tag{49}$$

since, as $p < 3/2$, we have $2/p - 1/3 > 1 + \theta = 1/p + 1/3$.

Finally, we have proven:

$$\|u_N - u_c\|_{L^p(K_0)} \le C(K,K_0,\varepsilon_0)|A|\left[\|\mathbb{M}_N[A] - \mathbb{M}_{eff}[A]\|_{\dot{H}^{-1}(\mathbb{R}^3)} + \left(\frac{a^3}{d^3}\right)^{1+\theta} + \|\mathbb{M}_{eff}\|^2_{L^\infty(\mathbb{R}^3)}\right].$$

This concludes the proof.

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Appendix A. Tools for the method of reflections

In this appendix, we give some technical tools that are involved in the method of reflections. We start with a representation formula generalizing the mean-value formula for harmonic functions.

Lemma A.1. *Suppose that $f \in L^1_{loc}(\mathbb{R}^3)$. Let D be a domain in $\mathbb{R}^3$ and $\Delta u = f$ in D. Then for arbitrary $x \in D$ and $r > 0$ such that $B(x,r) \subset D$ we have:*

$$u(x) = \fint_{B(x,r)} u(y)dy + \frac{1}{3}\int_0^r \left(\frac{\rho^4}{r^3} - \rho\right)\fint_{B(x,\rho)} f(y)dyd\rho.$$

Proof. This lemma must be part of the folklore. We give a proof for completeness. Let

$$\phi(r) = \fint_{\partial B(x,r)} u(y)d\sigma(y) = \frac{1}{4\pi}\int_{\partial B(0,1)} u(x+z)d\sigma(z).$$

After differentiation and integration by parts, we obtain:

$$\phi'(r) = \frac{1}{4\pi r^2}\int_{\partial B(x,r)} \partial_n u(y)d\sigma(y) = \frac{r}{3}\fint_{B(x,r)} f(y)dy.$$

Since $\phi(r) \to 0$ when $r \to 0$ we infer that:

$$\phi(r) = u(x) + \int_0^r \frac{\rho}{3}\fint_{B(x,\rho)} f(y)dyd\rho, \quad \text{if } B(x,r) \subset D.$$

Then, if $B(x,r) \subset D$ we have:

$$\fint_{B(x,r)} u(y)dy = \frac{1}{r^3}\int_0^r 3\rho^2\phi(\rho)d\rho.$$

Integrating by parts and applying the formula we derived above for $\phi(r)$ and $\phi'(r)$, we obtain:

$$\begin{aligned}\fint_{B(x,y)} u(y)dy &= \frac{1}{r^3}\left[r^3\phi(r) - \int_0^r \rho^3\phi'(\rho)d\rho\right]\\ &= \left(u(x) + \int_0^r \frac{\rho}{3}\fint_{B(x,\rho)} f(y)dyd\rho\right) - \frac{1}{r^3}\int_0^r \frac{\rho^4}{3}\fint_{B(x,\rho)} f(y)dyd\rho.\end{aligned}$$

This concludes the proof. □

Relying on this formula, we analyze the behavior of the recursive formula for the method of reflections (26). We recall that we consider here a set of centers of mass $(x_1, \dots, x_N)$ and parameters a, ε_0 such that $a^3/d^3 < \varepsilon_0$ where $d = \min_{i\neq j}|x_i - x_j|$. We include the recursive formula in the following more general framework. We assume we are given $V \in C^\infty(\mathbb{R}^3\setminus\{0\})$ homogeneous of degree -1 and we suppose that $\Delta V = Q$ where Q is harmonic in $\mathbb{R}^3 \setminus \{0\}$ and homogeneous with degree -3. We look then at quantities of the form

$$W_{l,\alpha} := a^3 \sum_{l\neq\lambda} m_\lambda \partial^\alpha V(x_l - x_\lambda) \quad \forall\, l = 1, \dots, N. \tag{50}$$

where $(m_1, \dots, m_N)$ are given and arbitrary and α is a multi-index in $\mathbb{N}^3$. The crucial result underlying the method of reflections is the following lemma:

Lemma A.2. *Let ε_0 small and $1 < q < \infty$. Then, there exists a constant $C(q, \varepsilon_0)$ such that the following properties hold true:*

(1) if $|\alpha| = 2$, then

$$\Big(\sum_{l=1}^N |W_{l,\alpha}|^q\Big)^{1/q} \leq C(q,\varepsilon_0)\big(\frac{a}{d}\big)^{3-3/q}\Big(\sum_{l=1}^N |m_l|^q\Big)^{1/q},$$

(2) if $|\alpha| = 1$ and we denote $F_l(x) := \sum_{l\neq\lambda} m_\lambda \nabla V(x - x_\lambda)$ on $B(x_l, 2a)$, there holds:

$$\sum_{l=1}^{N} \|\nabla F_l\|^2_{L^2(B(x_l,2a))} \le \frac{C(2,\varepsilon_0)}{d^3} \sum_{l=1}^{N} |m_l|^2. \tag{51}$$

Proof. We split the proof into two parts corresponding to the two items in the lemma.

Part 1. This part is the proof of the first statement of the above lemma. By definition of Q, we notice that $\Delta\partial^\alpha V = \partial^\alpha Q$ in $\mathbb{R}^3 \setminus \{0\}$. Hence according to Lemma A.1 and the fact that $\partial^\alpha Q$ is harmonic in $\mathbb{R}^3 \setminus \{0\}$, we obtain that

$$W_{l,\alpha} = W^1_{l,\alpha} + W^2_{l,\alpha} + W^3_{l,\alpha},$$

where

$$W^1_{l,\alpha} := a^3 \fint_{B(x_l,a)} \Big(\sum_{l\neq\lambda} \fint_{B(x_\lambda,d/2)} m_\lambda \partial^\alpha V(y-z)dz\Big)dy,$$

$$W^2_{l,\alpha} := \frac{a^3}{3}\int_0^a \left(\frac{r^4}{a^3} - r\right)\Big(\sum_{l\neq\lambda} \fint_{B(x_l,r)} m_\lambda \partial^\alpha Q(y-x_\lambda)dy\Big)dr$$

and

$$W^3_{l,\alpha} := \frac{a^3}{3}\fint_{B(x_l,a)} \Big(\sum_{l\neq\lambda}\int_0^{\frac{d}{2}} \left(\frac{8r^4}{d^3} - r\right)\fint_{B(x_\lambda,r)} m_\lambda \partial^\alpha Q(y-z)dzdr\Big)dy.$$

For the next computations, we introduce:

$$\Phi(x) := \sum_{l=1}^{N} m_l \mathbf{1}_{B(x_l,d/2)}(x) \tag{52}$$

and fix $q \in (1,\infty)$.

Step 1. In this part we deal with $W^1_{l,\alpha}$. By definition, we have that

$$W^1_{l,\alpha} = \frac{9}{2\pi^2 d^3}\int_{B(x_l,a)}\int_{\mathbb{R}^3\setminus B(x_l,d/2)} \Phi(z)\partial^\alpha V(y-z)dzdy.$$

In order to apply Calderón-Zygmund inequality, we split the above quantity into two parts: $W^1_{l,\alpha} = C_{l,\alpha} + R_{l,\alpha}$, where

$$C_{l,\alpha} := \frac{9}{2\pi^2 d^3}\int_{B(x_l,a)}\int_{\mathbb{R}^3\setminus B(y,d/2)} \Phi(z)\partial^\alpha V(y-z)dzdy,$$

$$R_{l,\alpha} := \frac{9}{2\pi^2 d^3}\int_{B(x_l,a)}\Big(\int_{\mathbb{R}^3\setminus B(x_l,d/2)} - \int_{\mathbb{R}^3\setminus B(y,d/2)}\Big)\Phi(z)\partial^\alpha V(y-z)dzdy.$$

We note that for any $y \in \mathbb{R}^3$,

$$\int_{\mathbb{R}^3\setminus B(y,d/2)} \Phi(z)\partial^\alpha V(y-z)dz = \int_{\mathbb{R}^3\setminus B(0,d/2)} \Phi(y-z)\partial^\alpha V(z)dz := G_\alpha(y),$$

which implies that

$$C_{l,\alpha} = \frac{9}{2\pi^2 d^3}\int_{B(x_l,a)} G_\alpha(y)dy.$$

Therefore, we have:

$$\big|C_{l,\alpha}\big| \leq Ca^{3-3/q}d^{-3}\|G_\alpha\|_{L^q(B(x_l,a))},$$

and

$$\Big(\sum_{l=1}^N \big|C_{l,\alpha}\big|^q\Big)^{1/q} \leq Ca^{3-3/q}d^{-3}\|G_\alpha\|_{L^q(\mathbb{R}^3)}.$$

On the other hand, by Calderón-Zygmund inequality we have:

$$\|G_\alpha\|_{L^q(\mathbb{R}^3)} \leq C(q)\|\Phi\|_{L^q(\mathbb{R}^3)} \leq C(q)d^{\frac{3}{q}}\Big(\sum_{l=1}^N |m_l|^q\Big)^{1/q}.$$

Hence we obtain that:

$$\Big(\sum_{l=1}^N \big|C_{l,\alpha}\big|^q\Big)^{1/q} \leq C(q)\big(\frac{a}{d}\big)^{3-3/q}\Big(\sum_{l=1}^N |m_l|^q\Big)^{1/q}. \tag{53}$$

Now we turn to deal with $R_{l,\alpha}$. At first, we notice that for any $l = 1, \dots, N$ and $x \in B(x_l, a)$, there holds:

$$B(x, d/2)\triangle B(x_l, d/2) \subset B(x, d/2 + a) \setminus B(x, d/2 - a),$$

(where $\triangle$ represents the symmetric difference between sets). Since $\partial^\alpha V$ is -3-homogeneous, this implies that for any $y \in B(x_l, a)$, we have:

$$\Big|\Big(\int_{\mathbb{R}^3\setminus B(x_l,d/2)} - \int_{\mathbb{R}^3\setminus B(y,d/2)}\Big)\Phi(z)\partial^\alpha V(y-z)dz\Big| \leq \int_{B(y,d+2a)\setminus B(y,d-2a)} \Big|\Phi(z)\frac{1}{|y-z|^3}\Big|dz.$$

We denote $\bar{G}_\alpha(y)$ the right-hand side of this inequality. Again by Hölder inequality, we obtain that

$$\big|R_{l,\alpha}\big| \leq Ca^{3-3/q}d^{-3}\|\bar{G}_\alpha\|_{L^q(B(x_l,a))},$$

and

$$\Big(\sum_{l=1}^N \big|R_{l,\alpha}\big|^q\Big)^{1/q} \leq Ca^{3-3/q}d^{-3}\|\bar{G}_\alpha\|_{L^q(\mathbb{R}^3)}.$$

On the other hand, by a standard Young inequality for convolution, we have

$$\begin{aligned}\|\bar{G}_\alpha\|_{L^q(\mathbb{R}^3)} &= \Big\|\int_{\mathbb{R}^3}|\Phi(\cdot - z)|\frac{\mathbf{1}_{B(0,d+2a)\setminus B(0,d-2a)}(z)}{|z|^3}dz\Big\|_{L^q(\mathbb{R}^3)} \\ &\leq C\ln\big(\frac{d+2a}{d-2a}\big)\|\Phi\|_{L^q(\mathbb{R}^3)} \leq C\ln\big(\frac{d+2a}{d-2a}\big)d^{3/q}\Big(\sum_{l=1}^N |m_l|^q\Big)^{1/q}.\end{aligned}$$

Therefore we obtain that

$$\Big(\sum_{l=1}^{N}\big|R_{l,\alpha}\big|^q\Big)^{1/q}\le C\ln\big(\frac{d+2a}{d-2a}\big)\big(\frac{a}{d}\big)^{3-3/q}\Big(\sum_{l=1}^{N}|m_l|^q\Big)^{1/q}. \tag{54}$$

By combing (53) and (54), we obtain finally that,

$$\Big(\sum_{l=1}^{N}\big|W^1_{l,\alpha}\big|^q\Big)^{1/q}\le C(q)\Big(1+\ln\big(\frac{d+2a}{d-2a}\big)\Big)\big(\frac{a}{d}\big)^{3-3/q}\Big(\sum_{l=1}^{N}|m_l|^q\Big)^{1/q}. \tag{55}$$

Step 2. Now we turn to handle $W^2_{l,\alpha}$. We recall that

$$W^2_{l,\alpha}=\frac{a^3}{3}\int_0^a\left(\frac{r^4}{a^3}-r\right)\big(\sum_{l\neq\lambda}⨍_{B(x_l,r)}m_\lambda\partial^\alpha Q(y-x_\lambda)dy\big)dr.$$

Since $\partial^\alpha Q$ is harmonic outside $B(0,a)$, we have for each $\lambda\neq l$ and $r<a$,

$$⨍_{B(x_l,r)}m_\lambda\partial^\alpha Q(y-x_\lambda)dy=⨍_{B(x_l,a)}⨍_{B(x_\lambda,d/2)}m_\lambda\partial^\alpha Q(y-z)dzdy,$$

which implies that

$$\begin{aligned}\sum_{l\neq\lambda}⨍_{B(x_l,r)}m_\lambda\partial^\alpha Q(y-x_\lambda)dy &= \sum_{l\neq\lambda}⨍_{B(x_l,a)}⨍_{B(x_\lambda,d/2)}m_\lambda\partial^\alpha Q(y-z)dzdy\\ &= \frac{9}{2\pi^2a^3d^3}\int_{B(x_l,a)}\int_{\mathbb{R}^3\setminus B(x_l,d/2)}\Phi(z)\partial^\alpha Q(y-z)dzdy.\end{aligned}$$

We also notice that

$$\big|\Phi(z)\partial^\alpha Q(y-z)\big|\le C|\Phi(z)|\frac{1}{|y-z|^5}.$$

Therefore we obtain that

$$\big|W^2_{l,\alpha}\big|\le Cd^{-3}\int_0^a rdr\int_{B(x_l,a)}\int_{\mathbb{R}^3\setminus B(x_l,d/2)}|\Phi(z)|\frac{1}{|y-z|^5}dzdy.$$

By a similar argument as before, we obtain that

$$\begin{aligned}\big|W^2_{l,\alpha}\big| &\le Ca^2d^{-3}\int_{B(x_l,a)}\int_{\mathbb{R}^3\setminus B(y,d/2-a)}|\Phi(z)|\frac{1}{|y-z|^5}dzdy\\ &\le a^{5-3/q}d^{-3}\Big\|\int_{\mathbb{R}^3\setminus B(0,d/2-a)}|\Phi(\cdot-z)|\frac{1}{|z|^5}dz\Big\|_{L^q(B(x_l,a))},\end{aligned}$$

which implies that, since $a^3/d^3<\varepsilon_0<<1$:

$$\begin{aligned}\Big(\sum_{l=1}\big|W^2_{l,\alpha}\big|^q\Big)^{1/q} &\le Ca^{5-3/q}d^{-3}\Big\|\int_{\mathbb{R}^3\setminus B(0,d/2-a)}|\Phi(\cdot-z)|\frac{1}{|z|^5}dz\Big\|_{L^q(\mathbb{R}^3)}\\ &\le C(q,\varepsilon_0)\big(\frac{a}{d}\big)^{5-\frac{3}{q}}\Big(\sum_{l=1}^{N}|m_l|^q\Big)^{1/q}.\end{aligned} \tag{56}$$

Step 3. At last we deal with $W^3_{l,\alpha}$. We recall that

$$W^3_{l,\alpha} = \frac{a^3}{3} \fint_{B(x_l,a)} \Big(\sum_{l\neq\lambda} \int_0^{\frac{d}{2}} \left(\frac{8r^4}{d^3} - r\right) \fint_{B(x_\lambda,r)} m_\lambda \partial^\alpha Q(y-z) dz dr\Big) dy.$$

Again by the fact that $\partial^\alpha Q$ is harmonic in $\mathbb{R}^3 \setminus \{0\}$ for any $|\alpha| = 2$, we obtain that for $y \in B(x_l, a)$, $l \neq \lambda$ and $r < d/2$:

$$\fint_{B(x_\lambda,r)} m_\lambda \partial^\alpha Q(y-z) dz = \fint_{B(x_\lambda,d/2)} m_\lambda \partial^\alpha Q(y-z) dz.$$

By a similar argument as in step 2, we obtain that

$$\begin{aligned} \left|W^3_{l,\alpha}\right| &\leq Cd^{-3} \int_{B(x_l,a)} \int_0^{d/2} r dr \sum_{l\neq\lambda} \int_{B(x_\lambda,d/2)} |\Phi(z)| \frac{1}{|y-z|^5} dz dy \\ &\leq Cd^{-1} \int_{B(x_l,a)} \sum_{l\neq\lambda} \int_{B(x_\lambda,d/2)} |\Phi(z)| \frac{1}{|y-z|^5} dz dy \\ &\leq Cd^{-1} \int_{B(x_l,a)} \int_{\mathbb{R}^3\setminus B(y,d/2-a)} |\Phi(z)| \frac{1}{|y-z|^5} dz dy \\ &\leq Ca^{3-3/q} d^{-1} \Big\| \int_{\mathbb{R}^3\setminus B(0,d/2-a)} |\Phi(\cdot - z)| \frac{1}{|z|^5} dz \Big\|_{L^q(B(x_l,a))}, \end{aligned}$$

and we conclude that:

$$\begin{aligned} \Big(\sum_{l=1} \left|W^3_{l,\alpha}\right|^q\Big)^{1/q} &\leq Ca^{3-3/q} d^{-1} \Big\| \int_{\mathbb{R}^3\setminus B(0,d/2)} |\Phi(\cdot - z)| \frac{1}{|z|^5} dz \Big\|_{L^q(\mathbb{R}^3)} \\ &\leq C(q) \Big(\frac{a}{d}\Big)^{3-\frac{3}{q}} \Big(\sum_{l=1}^N |m_l|^q\Big)^{1/q}. \end{aligned} \tag{57}$$

By combining (55), (56) and (57), we obtain the expected result since $a/d < \varepsilon_0 << 1$:

$$\begin{aligned} \Big(\sum_{l=1}^N \left|W_{l,\alpha}\right|^q\Big)^{1/q} &\leq \Big(\sum_{l=1}^N \left|W^1_{l,\alpha}\right|^q\Big)^{1/q} + \Big(\sum_{l=1}^N \left|W^2_{l,\alpha}\right|^q\Big)^{1/q} + \Big(\sum_{l=1}^N \left|W^3_{l,\alpha}\right|^q\Big)^{1/q} \\ &\leq C(q,\varepsilon_0) \Big(\frac{a}{d}\Big)^{3-\frac{3}{q}} \Big(\sum_{l=1}^N |m_l|^q\Big)^{1/q}. \end{aligned}$$

The first statement of the lemma is proved.

Part 2. In this part we give a proof for the second item. By definition of F_l and Lemma A.1, we have that for any $x \in B(x_l, 2a)$

$$F_l(x) = \sum_{l\neq\lambda} m_\lambda \fint_{B(x_\lambda,d/2)} \nabla V(x-y) dy + \frac{1}{3} \sum_{l\neq\lambda} m_\lambda \int_0^{\frac{d}{2}} \left(\frac{8r^4}{d^3} - r\right) \fint_{B(x_\lambda,r)} \nabla Q(x-y) dy dr.$$

According to the fact that Q is harmonic outside the origin, the second term on the right side of the above equation can be written as

$$\sum_{l\neq\lambda} m_\lambda \int_0^{\frac{d}{2}} \left(\frac{8r^4}{d^3} - r\right) \fint_{B(x_\lambda,r)} \nabla Q(x-y)dydr$$
$$= \sum_{l\neq\lambda} m_\lambda \int_0^{\frac{d}{2}} \left(\frac{8r^4}{d^3} - r\right) \fint_{B(x_\lambda,d/2)} \nabla Q(x-y)dydr$$
$$= \frac{33}{40\pi d} \sum_{l\neq\lambda} m_\lambda \int_{B(x_l,d/2)} \nabla Q(x-y)dy,$$

for any $x \in B(x_l, 2a)$. Therefore for any $x \in B(x_l, 2a)$, we have:

$$F_l(x) = \frac{6}{\pi d^3} \int_{\mathbb{R}^3\setminus B(x_l,d/2)} \Phi(y)\nabla V(x-y)dy + \frac{11}{40\pi d} \int_{\mathbb{R}^3\setminus B(x_l,d/2)} \Phi(y)\nabla Q(x-y)dy,$$

where Φ is defined in (52).

Now we start to prove (51). By the above argument and since the $B(x_l, 2a)$ are disjoint, we have

$$\sum_{l=1}^{N} \|\nabla F_l\|^2_{L^2(B(x_l,2a))} \leq C\frac{1}{d^6} \sum_{|\alpha|=2} \sum_{l=1}^{N} \| \int_{\mathbb{R}^3\setminus B(x_l,d/2)} \Phi(y)\partial^\alpha V(\cdot - y)dy\|^2_{L^2(B(x_l,2a))}$$
$$+ \frac{1}{d^2} \sum_{|\alpha|=2} \sum_{l=1}^{N} \| \int_{\mathbb{R}^3\setminus B(x_l,d/2)} \Phi(y)\partial^\alpha Q(\cdot - y)dy\|^2_{L^2(B(x_l,2a))}. \tag{58}$$

In order to control the right-hand side of the above inequality, we first notice that for each $l = 1, \dots, N$, any $|\alpha| = 2$ and $x \in B(x_l, 2a)$:

$$\int_{\mathbb{R}^3\setminus B(x_l,d/2)} \Phi(y)\partial^\alpha V(x-y)dy = \int_{\mathbb{R}^3\setminus B(0,d/2)} \Phi(x-y)\partial^\alpha V(y)dy$$
$$+\Big(\int_{\mathbb{R}^3\setminus B(x_l,d/2)} - \int_{\mathbb{R}^3\setminus B(x,d/2)} \Big)\Phi(y)\partial^\alpha V(x-y)dy.$$

By similar arguments as in **Part 1** of the proof, we obtain that

$$\sum_{|\alpha|=2} \sum_{l=1}^{N} \| \int_{\mathbb{R}^3\setminus B(0,d/2)} \Phi(\cdot - y)\partial^\alpha V(y)dy\|^2_{L^2(B(x_l,2a))} \leq Cd^3 \sum_{l=1}^{N} |m_l|^2$$

and

$$\sum_{|\alpha|=2}\sum_{l=1}^{N}\|\Big(\int_{\mathbb{R}^3\setminus B(x_l,d/2)}-\int_{\mathbb{R}^3\setminus B(x,d/2)}\Big)\Phi(y)\partial^\alpha V(x-y)dy\|^2_{L^2(B(x_l,2a))}$$
$$\leq C\left|\ln\Big(\frac{d+2a}{d-2a}\Big)\right|^2 d^3\sum_{l=1}^{N}|m_l|^2$$

Therefore we have

$$\sum_{|\alpha|=2}\sum_{l=1}^{N}\|\int_{\mathbb{R}^3\setminus B(x_l,d/2)}\Phi(y)\partial^\alpha V(\cdot-y)dy\|^2_{L^2(B(x_l,2a))} \leq Cd^3\left(1+\left|\ln\Big(\frac{d+2a}{d-2a}\right|^2\Big)\right)\sum_{l=1}^{N}|m_l|^2. \tag{59}$$

Now we turn to deal with the second term of the right side of (58). We first notice that for any $|\alpha|=2$, $l=1,\dots,N$ and $x\in B(x_l,2a)$, there holds:

$$\Big|\int_{\mathbb{R}^3\setminus B(x_l,d/2)}\Phi(y)\partial^\alpha Q(x-y)dy\Big|\leq C\int_{\mathbb{R}^3}|\Phi(y)|\frac{\mathbf{1}_{|x-y|>d/2-2a}}{|x-y|^5}dy,$$

Since $a^3/d^3<\varepsilon_0<<1$ we obtain via a standard Young inequality for convolutions that

$$\sum_{|\alpha|=2}\sum_{l=1}^{N}\|\int_{\mathbb{R}^3\setminus B(x_l,d/2)}\Phi(y)\partial^\alpha Q(\cdot-y)dy\|^2_{L^2(B(x_l,2a))}\leq C(\varepsilon_0)d^{-1}\sum_{l=1}^{N}|m_l|^2. \tag{60}$$

Finally, combining (58), (59) and (60) and remarking that $d\leq diam(K)$, we have that

$$\sum_{l=1}^{N}\|\nabla F_l\|^2_{L^2(B(x_l,2a))}\leq C(\varepsilon_0)d^{-3}\sum_{l=1}^{N}|m_l|^2.$$

This ends the proof of the second item and the proof of the lemma. □

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