ON THE DERIVATION OF A STOKES-BRINKMAN PROBLEM FROM STOKES EQUATIONS AROUND A RANDOM ARRAY OF MOVING SPHERES

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Abstract. We consider the Stokes system in $\mathbb{R}^3$, deprived of N spheres of radius $1/N$, completed by constant boundary conditions on the spheres. We assume that the centers of the spheres and the boundary conditions are given randomly and we compute the asymptotic behavior of solutions when the parameter N diverges. Under the assumption that the distribution of spheres/centers is chaotic, we prove convergence in mean to the solution of a Stokes-Brinkman problem.

Contents

1. Introduction

This paper is a contribution to a rigorous justification of mesoscopic models for the motion of a cloud of solid particles in a viscous fluid. As explained in [7], the modeling of particle suspensions can borrow to different areas of partial differential equations. If the cloud contains few particles, the behavior of particles can be modeled by a finite dimensional system and the coupling with the fluid equations yields a fluid/solid problem similar to the ones studied in [5, 6, 11, 23] for example. If the number of particle increases, a description of the particle phase *via* its individuals seems irrelevant. Depending on the volume fraction of the particle phase it is then necessary to turn to a kinetic/fluid description (as in [2] or [3]) or a multiphase description (see [12]).

In the case of a kinetic/fluid description, a system – that we can find in references – is the following Vlasov–Navier-Stokes system:

$$\begin{aligned} \partial_t f + v\cdot\nabla_x f + 6\pi\ \mathrm{div}_v[(u-v)f] &= 0\,,\\ (\partial_t u + u\cdot\nabla_x u) &= \Delta_x u - \nabla_x p - 6\pi\int_{\mathbb{R}^3} f(u-v)\ \mathrm{d}v\,,\\ \mathrm{div}_x u &= 0\,. \end{aligned}$$

Here we introduce $f:(t,x,v)\in[0,\infty)\times\mathbb{R}^3\times\mathbb{R}^3\to[0,\infty)$ the particle distribution function which counts the proportion of particles at time t which are in position $x\in\mathbb{R}^3$ and have velocity $v\in\mathbb{R}^3$. This unknown encodes the cloud behavior. We emphasize that v is a parameter of f, hence the notations with indices to express with respect to which variable we differentiate. The two other unknowns (u,p) represent respectively the fluid velocity-field and pressure. One recognizes in the two last equations Navier-Stokes like equations. For simplicity,

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we do not include physical parameters such as the fluid density and viscosity. A particular feature of this model is the supplementary term

$$6\pi \int_{\mathbb{R}^3} f(u-v)\,\mathrm{d}v\,, \tag{1.1}$$

that appears on the right-hand side of the momentum equation. It is supposed to model the exchange of momentum between the solid phase and the fluid. It can be justified with the following formal reasoning. Assume that the cloud is made of N identical spheres of radius $1/N$. If the particles are sufficiently spaced, they interact with the fluid as if they where alone: at its own scale, the particle i moves with its velocity v_i in a viscous fluid whose velocity at infinity is $u(h_i)$. Stokes' law entails that fluid viscosity is responsible of the drag force:

$$F_i = \frac{6\pi}{N}(v_i - u(h_i)).$$

This term corresponds to the forcing term in the Vlasov equation and the corresponding term (1.1) in the Navier-Stokes system is obtained by assuming that the particle forces can be superposed.

We are interested here in a rigorous approach to the above formal reasoning. This supposes to start from the fluid/solid problem, where the particle dynamics equations are solved individually, and let the number of particles diverges with their radius and density given by a suitable scaling. This question mixes large particle system problems (justification of the Vlasov equations starting from a system of ODEs) with fluid homogenization issues (computing a macroscopic equation for the fluid unknowns). The full problem seeming still out of reach now, we focus here on the fluid homogenization part. Namely, one assumes that the particle behavior is given and wants to compute the new term in the fluid equation which takes into account the influence of the particles. Since this term is due to fluid viscosity, we restrict to the Stokes system (*i.e.* the system obtained by neglecting the full time derivative on the left hand side of the momentum fluid equation). Then, the problem reduces to homogenizing the Stokes problem in a perforated domain with non-zero boundary conditions (mimicking the particle translation). This particular homogenization problem has been the subject of recent publications (see [8, 14, 16, 19]). Therein, the limit Stokes system including the Brinkman term (1.1) is obtained under specific dilution assumption of the particle phase. One further step toward tackling the time-dependent problem is then to discuss whether the set of favorable configurations – *i.e.* such that the Brinkman term (1.1) appears in the limit – is sufficiently large. To this end, we propose here to derive the Stokes-Brinkman problem via a Liouville approach in the spirit of [20]. More precisely, we first pick at random N identical spherical particles/obstacles of radius $1/N$, each of them being characterized by its center of mass and its velocity, under the constraint that particles do not intersect each other. We assume that the cloud of particles lies within a bounded open subset Ω_0 of $\mathbb{R}^3$. We then consider a fluid occupying the whole space $\mathbb{R}^3$ deprived of these particles and satisfying a stationary Stokes equation with Dirichlet boundary condition at the boundary of each particle given by its velocity. Our aim is to rigorously derive the Stokes-Brinkman equation as an effective equation of the above problem in the limit $N \to \infty$.

Let us describe the problem in details. To begin with, fix $N \in \mathbb{N}^*$ arbitrary large and consider the experiment of dropping randomly N spheres of radius $1/N$ in the whole space $\mathbb{R}^3$. Since the radius of the spheres is very small in comparison with their number (note that the volume fraction occupied by the spheres is typically of size $1/N^2$), we adapt a model that is classical for large point-particle systems. We denote

$$\mathcal{O}^N := \Big\{ ((X_1^N, V_1^N), \dots, (X_N^N, V_N^N)) \in [\mathbb{R}^3 \times \mathbb{R}^3]^N \text{ s.t. } |X_i^N - X_j^N| > \frac{2}{N} \quad \forall\, i \neq j \Big\}.$$

This represents the set of admissible configurations for the centers of mass $\mathbf{X}^N = (X_1^N, \dots, X_N^N)$ and velocities $\mathbf{V}^N = (V_1^N, \dots, V_N^N)$. In what follows, we also denote $Z_i = (X_i, V_i)$ the state variable for the particle i and keep bold symbols for N-component entities. For instance, we denote $\mathbf{Z}^N = ((X_1^N, V_1^N), \dots, (X_N^N, V_N^N)) \in \mathcal{O}^N$ a configuration.

The configuration of particles $\mathbf{Z}^N$ will be chosen at random under some law $F^N \in \mathcal{P}(\mathcal{O}^N)$, where we denote by $\mathcal{P}(E)$ the space of probability measures on E. We assume that this probability measure is absolutely continuous w.r.t. the Lebesgue measure and also denote by F^N its density. Moreover, since the particles are indistinguishable, we shall assume that $\mathbf{Z}^N$ is an exchangeable random variable, which means that its law F^N is symmetric, that is, for any permutation $\sigma \in \mathfrak{S}_N$ there holds

$$F^N(Z_1^N, \dots, Z_N^N) = F^N(Z_{\sigma(1)}^N, \dots, Z_{\sigma(N)}^N), \quad \forall\, \mathbf{Z}^N \in \mathcal{O}^N.$$

Given a configuration $\mathbf{Z}^N = ((X_1^N, V_1^N), \dots, (X_N^N, V_N^N)) \in \mathcal{O}^N$ we introduce the perforated domain:

$$\mathcal{F}^N = \mathbb{R}^3 \setminus \bigcup_{i=1}^N \overline{B_i^N}, \qquad \text{where } B_i^N = B(X_i^N, \tfrac{1}{N}) \quad \forall\, i = 1, \dots, N,$$

and consider the following Stokes problem:

$$\left\{ \begin{array}{rcl} -\Delta u + \nabla p & = & 0 \\ \operatorname{div} u & = & 0 \end{array} \right. \quad \text{in } \mathcal{F}^N, \tag{1.2}$$

with boundary conditions

$$\begin{array}{rcl} u(x) & = & V_i^N \quad \text{on } \partial B_i^N \text{ for } i = 1, \dots, N, \\ \lim\limits_{|x| \to \infty} |u(x)| & = & 0. \end{array} \tag{1.3}$$

We obtain a stationary exterior problem in 3 dimensions. Such systems are extensively studied in [10, Section V] where it is proven for instance that there exists a unique solution (u, p) to (1.2)-(1.3). We may then construct:

$$u[\mathbf{Z}^N](x) = \left\{ \begin{array}{ll} u(x), & \text{if } x \in \mathcal{F}^N \\ V_i^N, & \text{if } x \in B_i^N \text{ for } i = 1, \dots, N. \end{array} \right.$$

The above reference on the exterior problem entails that $u[\mathbf{Z}^N] \in \dot{H}^1(\mathbb{R}^3)$ (where we denote $\dot{H}^1(\mathbb{R}^3)$ the closure of $C_c^\infty(\mathbb{R}^3)$ for the L^2-norm of the gradient). Therefore, we construct the mapping

$$\begin{array}{rccc} U_N : & \mathcal{O}^N & \longrightarrow & \dot{H}^1(\mathbb{R}^3) \\ & \mathbf{Z}^N & \longmapsto & u[\mathbf{Z}^N] \end{array} \tag{1.4}$$

as a random variable on $\mathcal{O}^N$ endowed with the probability measure F^N.

At first in [8], it is shown that, for a given sequence $\mathbf{Z}^N$ satisfying some conditions and with prescribed asymptotic behavior when $N \to \infty$, the associated solution to (1.2)-(1.3) converge to a solution to the Stokes-Brinkman problem:

$$\left\{ \begin{array}{rcl} -\Delta \tilde{u} + \nabla \tilde{p} + 6\pi\rho\tilde{u} & = & 6\pi j \\ \operatorname{div} \tilde{u} & = & 0 \end{array} \right. \quad \text{in } \mathbb{R}^3, \tag{1.5}$$

with vanishing condition at infinity

$$\lim_{|x| \to \infty} |\tilde{u}(x)| = 0. \tag{1.6}$$

In this system the flux j and density ρ are related to the asymptotic behavior of the $\mathbf{Z}^N$. In this paper, we compute the flux j and density ρ depending on the asymptotic behavior of the law F^N in order that the expectation of U_N converge in a suitable sense to the same Stokes-Brinkman problem. As we recall in the beginning of Section 4, this system is well-posed for positive $\rho \in L^{3/2}(\mathbb{R}^3)$ and $j \in L^{6/5}(\mathbb{R}^3)$.

1.1. **Main result.** Our main result requires some conditions on the sequence of symmetric probability measures $(F^N)_{N \in \mathbb{N}^*}$ on $\mathcal{O}^N$. We introduce, for an integer $m \le N$, the m-marginal distributions of F^N by

$$F_m^N(\mathbf{z}) = \int_{\mathbb{R}^{6(N-m)}} \mathbf{1}_{(\mathbf{z}, \mathbf{z}') \in \mathcal{O}^N} F^N(\mathbf{z}, \mathbf{z}') \mathrm{d}\mathbf{z}' \qquad \forall\, \mathbf{z} \in \mathcal{O}^m.$$

Here we observe that such marginals are constructed by remarking that, if we split an $N-$particle distribution by giving the m first particle state $\mathbf{z}$ and the remaining $(N-m)$ particle state $\mathbf{z}'$ we must require that $\mathbf{z} \in \mathcal{O}^m$ in order that $(\mathbf{z}, \mathbf{z}') \in \mathcal{O}^N$ be possible. We apply here again with small letters the convention that $z_i \in \mathbb{R}^6$ splits into $z_i = (x_i, v_i)$ and that bold symbols encode vectors of unknowns x, v or z.

We are now able to state our main assumptions. Let $(\mathbf{Z}^N)_{N \in \mathbb{N}^*}$ be a sequence of exchangeable $\mathcal{O}^N$-valued random variables, and let $(F^N)_{N \in \mathbb{N}^*}$ be the sequence of their associated laws, that is, symmetric probability measures on $\mathcal{O}^N$.

Assumption A1. We assume that $(F^N)_{N \in \mathbb{N}^*}$ are distribution functions, that is belong to $L^1(\mathcal{O}^N)$, and satisfy the following properties:

(0) $\mathrm{Supp}(F^N) \subset (\Omega_0 \times \mathbb{R}^3)^N$, for some bounded open $\Omega_0 \subset \mathbb{R}^3$ and any $N \in \mathbb{N}^*$.

(1) There exists a constant $C_1 \ge 1$ such that for any $N \in \mathbb{N}^*$ and $1 \le m \le N$

$$\|F_m^N\|_{L_x^\infty L_v^1(\mathcal{O}^m)} := \sup_{\mathbf{x} \in \mathbb{R}^{3m}} \int_{\mathbb{R}^{3m}} \mathbf{1}_{\mathbf{z} \in \mathcal{O}^m} F_m^N(\mathbf{z}) \, \mathrm{d}\mathbf{v} \le (C_1)^m.$$

(2) There exists $k_0 \geq 5$ and a constant $C_2 > 0$ such that

$$\sup_{N\in\mathbb{N}^*} \||z_1|^{k_0} F_1^N\|_{L^1_x L^1_v(\mathbb{R}^3\times\mathbb{R}^3)} = \sup_{N\in\mathbb{N}^*} \int_{\mathbb{R}^3\times\mathbb{R}^3} |z_1|^{k_0} F_1^N(z_1)\,\mathrm{d}z_1 \leq C_2.$$

(3) There exists a constant $C_3 > 0$ such that

$$\sup_{N\in\mathbb{N}^*} \||v_1| F_2^N\|_{L^\infty_x L^1_v(\mathcal{O}^2)} = \sup_{N\in\mathbb{N}^*} \sup_{x_1,x_2} \int_{\mathbb{R}^6} \mathbf{1}_{(z_1,z_2)\in\mathcal{O}^2} |v_1| F_2^N(z_1,z_2)\,\mathrm{d}v_1\mathrm{d}v_2 \leq C_3.$$

In this set of assumptions, (2) corresponds to the classical assumption that the law has a sufficient number of bounded moments; (1) would be satisfied in particular by tensorized laws; (0) is reminiscent of the fact that the cloud occupies the bounded region Ω_0 and (3) shall enable to control the interactions between close particles through the flow.

Given a sequence $(\mathbf{Z}^N)_{N\in\mathbb{N}^*}$ of exchangeable random variables on $\mathcal{O}^N$, we define the associated empirical measure by

$$\mu^N[\mathbf{Z}^N] := \frac{1}{N}\sum_{i=1}^N \delta_{Z_i^N}, \tag{1.7}$$

as well as the empirical density and the empirical flux respectively by

$$\rho^N[\mathbf{Z}^N] = \rho^N[\mathbf{X}^N] := \frac{1}{N}\sum_{i=1}^N \delta_{X_i^N}, \quad j^N[\mathbf{Z}^N] := \frac{1}{N}\sum_{i=1}^N V_i^N \delta_{X_i^N}. \tag{1.8}$$

The first formula defines a standard probability measure while the second one is a vectorial measure on $\mathbb{R}^3$.

We now state our assumptions concerning the asymptotic behavior of the sequence of configurations.

Assumption A2. Under Assumption A1, we suppose that there is a probability measure f on $\mathbb{R}^3\times\mathbb{R}^3$ with support on $\Omega_0\times\mathbb{R}^3$ such that, defining the probability measure $\rho(\mathrm{d}x) = \int_{\mathbb{R}^3} f(\mathrm{d}x,\mathrm{d}v)$ and the vectorial measure $j(\mathrm{d}x) := \int_{\mathbb{R}^3} v f(\mathrm{d}x,\mathrm{d}v)$ (both with support on Ω_0), we have:

(i) $\lim_{N\to\infty} \mathbb{E}\left[W_1(\rho^N[\mathbf{Z}^N],\rho)\right] = 0$;

(ii) $\lim_{N\to\infty} \mathbb{E}\left[\|j^N[\mathbf{Z}^N] - j\|_{[C^{0,1}_b(\mathbb{R}^3)]^*}\right] = 0.$

Remark 1. Given the random variable $\mathbf{Z}^N$ with law F^N, we can consider the random variable $\mathbf{X}^N$ on $\mathcal{O}^N_x := \{(X_1^N,\dots,X_N^N) \in \mathbb{R}^{3N} \mid |X_i^N - X_j^N| > \frac{2}{N}\ \forall i\neq j\}$ which has a symmetric law $R^N \in \mathcal{P}(\mathcal{O}^N_x)$, given by $R^N(\mathrm{d}\mathbf{x}^N) = \int_{\mathbb{R}^{3N}} F^N(\mathrm{d}\mathbf{x}^N,\mathrm{d}\mathbf{v}^N)$. Point (i) in Assumption A2 is equivalent to the fact that the sequence $(R^N)_{N\in\mathbb{N}^*}$ is ρ-chaotic (roughly speaking that R^N is asymptoticly i.i.d. with law ρ, see Definition 2.1) thanks to e.g. [13].

Remark 2. We will be interested in conditions on the sequence $(F^N)_{N\in\mathbb{N}^*}$ in order to ensure the convergences of Assumption A2. In particular we will show in Lemma 2.3 that if the sequence $(F^N)_{N\in\mathbb{N}^*}$ is f-chaotic (see Definition 2.1) then it satisfies Assumption A2. (But clearly this is not a necessary condition.)

With these notations, our main theorem reads:

Theorem 1.1. *Let $f\in L^1(\mathbb{R}^3\times\mathbb{R}^3)$ be a probability measure having support in $\Omega_0\times\mathbb{R}^3$ and define $\rho(x) = \int_{\mathbb{R}^3} f(x,v)\,\mathrm{d}v$ and $j(x) = \int_{\mathbb{R}^3} v f(x,v)\,\mathrm{d}v$. Assume that $\rho\in L^3(\Omega_0)$ and $j\in L^{6/5}(\Omega_0)$ so that there exists a unique solution $(u,p)\in \dot H^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$ to the Stokes-Brinkman problem (1.5)-(1.6) associated to ρ and j. Consider a sequence of exchangeable random variables $(\mathbf{Z}^N)_{N\in\mathbb{N}^*}$ on $\mathcal{O}^N$ and their associated symmetric laws $(F^N)_{N\in\mathbb{N}^*}$ satisfying Assumption A1.*

Then, given $\alpha\in(2/3,1)$ and for N large enough, the map U_N given by (1.4) satisfies:

$$\mathbb{E}\left[\|U_N[\mathbf{Z}^N] - u\|_{L^2_{\mathrm{loc}}(\mathbb{R}^3)}\right] \lesssim \mathbb{E}\left[W_1(\rho^N[\mathbf{Z}^N],\rho)\right]^{\frac{1}{39}} + \mathbb{E}\left[\|j^N[\mathbf{Z}^N] - j\|_{[C^{0,1}_b(\mathbb{R}^3)]^*}\right]^{\frac{1}{3}} + N^{-e_1(\alpha)}, \tag{1.9}$$

where $e_1(\alpha) = \min(\frac{1-\alpha}{65}, \frac{(3\alpha-2)}{2})$.

As a consequence, if $(F^N)_{N\in\mathbb{N}^*}$ satisfies moreover Assumption A2 (i)-(ii), then

$$\lim_{N\to\infty} \mathbb{E}\left[\|U_N[\mathbf{Z}^N] - u\|_{L^2_{\mathrm{loc}}(\mathbb{R}^3)}\right] = 0.$$

A key-point in the result of this theorem is that the right-hand side of (1.9) depends on powers of $\mathbb{E}\left[W_1(\rho^N[\mathbf{Z}^N],\rho)\right]$ and $\mathbb{E}\left[\|j^N[\mathbf{Z}^N]-j\|_{[C_b^{0,1}(\mathbb{R}^3)]^*}\right]$, and on a residual power of N (depending only on the parameter α). We claim that this structure is universal but the explicit values of our exponents need not be optimal in all contexts, furthermore it is also possible to obtain a L^p version of estimate (1.9) with different exponents, under the condition that $W^{2,p}$ embeds into some Hölder space.

A further result of our study (see Section 3), is that, with the assumptions of Theorem 1.1, $\mathbb{E}[U_N[\mathbf{Z}^N]]$ defines a bounded sequence in $\dot{H}^1(\mathbb{R}^3)$. Theorem 1.1 then implies that this sequence converges (at least weakly in $\dot{H}^1(\mathbb{R}^3)$) to the solution to the Stokes-Brinkmann problem with the corresponding flux j and density ρ. This consequence is yet another hint that the Stokes-Brinkman problem (1.5)-(1.6) is indeed the right macroscopic model to compute the behavior of a viscous fluid in presence of a cloud of moving particles under the asymptotic convergences of Assumption A2.

To show one application of the previous theorem, we shall construct an explicit example of probability measure on $\mathcal{O}^N$ satisfying the assumptions of Theorem 1.1 and for which we obtain a quantitative estimate of the convergence (1.9).

Corollary 1.2. *Let $f\in L^1(\mathbb{R}^3\times\mathbb{R}^3)$ be a probability measure satisfying the hypotheses of Theorem 1.1 and such that the associated density $\rho\in L^\infty(\Omega_0)$ and $\int_{\Omega_0\times\mathbb{R}^3}|v|^k f(\mathrm{d}z)$ for some $k\geq 5$. Then we can construct a sequence of symmetric probability measures $(F^N)_{N\in\mathbb{N}^*}$ on $\mathcal{O}^N$ satisfying Assumptions A1 and A2, and for which there holds*

$$\mathbb{E}\left[\|U_N[\mathbf{Z}^N]-u\|_{L^2_{\mathrm{loc}}(\mathbb{R}^3)}\right]\lesssim N^{-\frac{1}{117}}+N^{-e_1(\alpha)}.$$

1.2. **Overview of the proof.** The proof of Theorem 1.1 faces several difficulties. First, for fixed N, we must identify a sufficiently large set of data $\mathbf{Z}^N$ for which the solution $U_N[\mathbf{Z}^N]$ to the Stokes problem in the punctured domain is close to the solution to the Stokes-Brinkman problem. In comparison with [8], a key-difficulty is to have a quantified estimate at-hand. A second difficulty is that, since the velocities V_i^N that we impose on the particles are arbitrary, the solution to the Stokes problem may diverge in $\dot{H}^1(\mathbb{R}^3)$ when two particles become close. It is then necessary to obtain a bound on the solution to the Stokes problem associated with these configurations in order to ensure that they won't perturb the computation of the limit in mean.

Having in mind these two important difficulties, we propose an approach that is divided into five steps that we explain in more details below:

- As a first step, we prove in Section 2.1 some estimates associated to the convergence of the sequence of configurations (the random variables $(\mathbf{Z}^N)_N$ and their laws $(F^N)_N$) with respect to the expected limit (the marginals ρ and j of the distribution f).
- We then identify some "concentrated configurations" and prove that they are negligible in the asymptotic limit $N\to\infty$. These configurations correspond to $\mathbf{Z}^N\in\mathcal{O}^N$ such that there exists a couple of particles too close to each other or that there exist too many particles in a same cell of small volume. This is done in Section 2.2.
- Furthermore, we compute uniform estimates satisfied by the map $U_N[\mathbf{Z}^N]$. We obtain simultaneously that:
  - the mean of $U_N[\mathbf{Z}^N]$ is well-defined and uniformly bounded in $\dot{H}^1(\mathbb{R}^3)$;
  - the weight of contribution of the concentrated configurations vanishes when $N\to\infty$.

  This enables to get rid of concentrated configurations in the asymptotic description of U_N. This step is treated in Section 3.
- In a further step, developed in Section 4, we prove a mean-field result for non-concentrated configurations which is the cornerstone of our proof. We combine here the duality method of [19] with covering arguments of [14]. In comparison with these previous references, we consider in this paper an unbounded container. So, these arguments need to be adapted carefully.
- Finally, in the last step presented in Section 5, we gather previous estimates together in order to obtain Theorem 1.1. Furthermore, we construct a particular example of sequence of probability measures $(F^N)_N$ in order to obtain Corollary 1.2.

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2. Properties of the sequence of configurations

In this section we gather some properties of the sequence of configurations $(\mathbf{Z}^N)_{N\in\mathbb{N}^*}$ on $\mathcal{O}^N$ under the sequence of associated laws $(F^N)_{N\in\mathbb{N}^*}$ satisfying Assumptions A1. We recall that

$$\mathcal{O}^N := \Big\{\mathbf{Z}^N \in (\mathbb{R}^3\times\mathbb{R}^3)^N \mid |X_i - X_j| > \frac{2}{N} \quad \forall i\neq j\Big\}, \tag{2.1}$$

where hereafter we shall use the Assumption A1-(0) saying that $\mathrm{Supp}(F^N)\subset \Omega_0\times\mathbb{R}^3$ for some bounded open set $\Omega_0\subset\mathbb{R}^3$, and where we denote

$$\begin{aligned}
&\mathbf{Z}^N = (Z_1,\dots,Z_N) \in (\mathbb{R}^3\times\mathbb{R}^3)^N,\\
&\mathbf{X}^N = (X_1,\dots,X_N)\in\mathbb{R}^{3N}, \quad \mathbf{V}^N = (V_1,\dots,V_N)\in\mathbb{R}^{3N},\\
&Z_i = (X_i,V_i)\in\mathbb{R}^3\times\mathbb{R}^3.
\end{aligned}$$

We shall denote by $\mathcal{O}^N_x$ the projection of the space of configurations $\mathcal{O}^N$ onto the $\mathbf{X}^N$-variables, more precisely

$$\mathcal{O}^N_x := \Big\{(X_1,\dots,X_N)\in\mathbb{R}^{3N} \mid |X_i - X_j| > \frac{2}{N} \quad \forall i\neq j\Big\},$$

in such a way that $\mathcal{O}^N \simeq \mathcal{O}^N_x\times\mathbb{R}^{3N}$.

In the first part of this section, we focus on the convergence of the family of measures $(\rho^N[\mathbf{Z}^N])_{N\in\mathbb{N}^*}$ and $(j^N[\mathbf{Z}^N])_{N\in\mathbb{N}^*}$ seen as random variables. As mentioned in the introduction, we metrize the convergence of measures on $\mathbb{R}^3$ by two different topologies: either we see (by restriction) vectorial measures as bounded linear forms on Hölder spaces:

$$C^{0,\theta}_b(\mathbb{R}^3) := \left\{\varphi\in C(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)\,,\ \text{s.t.}\ \sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|^\theta}<\infty\right\},$$

or we use the (Monge-Kantorovich-)Wasserstein W_1-distance on probability measures. Hereafter, the 1-Wasserstein distance $W_1(f,g)$, with f and g probability measures on $\mathbb{R}^3\times\mathbb{R}^3$, is defined by

$$W_1(g,f) := \inf_{\pi\in\Pi(g,f)}\int_{(\mathbb{R}^3\times\mathbb{R}^3)^2}|z-z'|\mathrm{d}\pi(z,z') = \sup_{[\psi]_{\mathrm{Lip}(\mathbb{R}^3\times\mathbb{R}^3)}\le 1}\int_{\mathbb{R}^3\times\mathbb{R}^3}\psi(z)\,(g(\mathrm{d}z)-f(\mathrm{d}z))\,, \tag{2.2}$$

with $\Pi(g,f)$ being the set of probability measures on $(\mathbb{R}^3\times\mathbb{R}^3)^2$ whose first marginal equals g and second marginal f, and $[\cdot]_{\mathrm{Lip}(\mathbb{R}^3\times\mathbb{R}^3)}$ denotes the Lipschitz semi-norm (see e.g. [24]). Correspondingly, $[\cdot]_{C^{0,\theta}}$ stands for the $C^{0,\theta}$ semi-norm.

In the second part of this section, we shall measure the weights of configurations in which the particles are concentrated, meaning that the minimal distance between two particles is small or that there are too many particles in a small subset of $\mathbb{R}^3$.

2.1. **On the convergence Assumption A2.**

Let us describe some properties concerning the asymptotic convergence of the data, where we always assume that Assumption A1 is in force. We shall first obtain some estimates for different metrics concerning the convergences of Assumption A2, and then we shall give a sufficient condition on the sequence $(F^N)_{N\in\mathbb{N}^*}$ to satisfy Assumption A2.

We recall below the notion of chaoticity for a sequence of probability measures, see [18, 22].

Definition 2.1. Let $E\subset\mathbb{R}^d$. Consider a sequence $(\mathbf{Y}^N)_{N\in\mathbb{N}^*}$ of exchangeable random variables on E^N and the associated sequence of laws $(\pi^N)_{N\in\mathbb{N}^*}$, that are symmetric probability measures on E^N. We say that $(\pi^N)_{N\in\mathbb{N}^*}$ (or that $(\mathbf{Y}^N)_{N\in\mathbb{N}^*}$) is π-chaotic, for some probability measure π on E, if one of the following equivalent conditions is fulfilled:

(a) π^N_m converges to $\pi^{\otimes m}$ weakly in $\mathcal{P}(E^m)$ as $N\to\infty$ for any fixed $m\ge 1$ (or some $m\ge 2$);
(b) the $\mathcal{P}(E)$-valued random variable $\mu^N[\mathbf{Y}^N]$ converges in law to π as $N\to\infty$.

Here π^N_m denotes the m-marginal of π^N given by $\pi^N_m(\mathrm{d}z_1,\dots,\mathrm{d}z_m) := \int_{E^{N-m}}\pi^N(\mathrm{d}z_1,\dots,\mathrm{d}z_m,\mathrm{d}z_{m+1},\dots,\mathrm{d}z_N)$, and $\mu^N[\mathbf{Y}^N] = \frac{1}{N}\sum_{i=1}^N\delta_{Y_i^N}$ is the empirical measure associated to $\mathbf{Y}^N$.

We remark that [13] obtain a quantitave version of the above equivalence. More precisely, assuming that π^N_1 possesses a finite moment of order $k>1$, $(\pi^N)_{N\in\mathbb{N}^*}$ is π-chaotic is equivalent to

(a') $\lim\limits_{N\to\infty} W_1(\pi^N_m,\pi^{\otimes m}) = 0$ for any fixed $m\ge 1$ (or some $m\ge 2$);

(b') $\lim_{N\to\infty} \mathbb{E}\left[W_1(\mu^N[\mathbf{Y}^N],\pi)\right] = 0$;

with a quantitative estimate in N for the equivalence between (a') and (b'). As a consequence of this, and arguing similarly for the case of finite vectorial measures (more precisely for finite signed measures, corresponding to each component of $j^N[\mathbf{Z}^N]$ and j), we hence remark that Assumption A2 is equivalent to

(i') the random variable $\rho^N[\mathbf{Z}^N]$ converges in law to ρ as $N\to\infty$;
(ii') the random variable $j^N[\mathbf{Z}^N]$ converges in law to j as $N\to\infty$.

We now give some estimates concerning different metrics. For any $k>0$ and any probability measure $f\in\mathcal{P}(\mathbb{R}^3\times\mathbb{R}^3)$ with support on $\Omega_0\times\mathbb{R}^3$, we denote its moment of order $k>0$ by

$$M_k(f) := \int_{\Omega_0\times\mathbb{R}^3} (1+|v|^2)^{k/2}\, f(\mathrm{d}x,\mathrm{d}v).$$

We remark that $M_k(f)\geq 1$ for any $k>0$ and $k\mapsto M_k(f)$ is non-decreasing. On the other hand, under the Assumption A1-(2), we have a uniform bound for $(M_{k_0}(F^N))_{N\in\mathbb{N}^*}$. So, below, we focus on probability measures with bounded k_0-momentum *i.e.*:

$$\mathcal{B}_{k_0}(C_2) := \{f\in\mathcal{P}(\mathbb{R}^3\times\mathbb{R}^3) \text{ s.t. } \mathrm{Supp}(f)\subset\Omega_0\times\mathbb{R}^3 \text{ and } M_{k_0}(f)\leq C_2\}$$

where $k_0\in[1,\infty)$ and $C_2\geq 1$ are fixed by Assumption A1-(2). Standard arguments show that this set is closed w.r.t. the weak topology on $\mathcal{P}(\mathbb{R}^3\times\mathbb{R}^3)$.

Lemma 2.2. *Let $f,g\in\mathcal{B}_{k_0}(C_2)$ and define $\rho_f=\int_{\mathbb{R}^3} f(\cdot,\mathrm{d}v)$, $\rho_g=\int_{\mathbb{R}^3} g(\cdot,\mathrm{d}v)$, $j_f=\int_{\mathbb{R}^3} vf(\cdot,\mathrm{d}v)$ and $j_g=\int_{\mathbb{R}^3} vg(\cdot,\mathrm{d}v)$. Given $k>0$ we denote $\mathcal{M}_k := M_k(f)+M_k(g)$ and $K_0>0$ a constant depending on Ω_0.*

(1) For any $\theta\in(0,1)$ there holds

$$\|\rho_f-\rho_g\|_{(C_b^{0,\theta}(\mathbb{R}^3))^*} \leq K_0 W_1(\rho_f,\rho_g)^{\frac{\theta}{\theta+1}} \leq K_0 W_1(f,g)^{\frac{\theta}{\theta+1}}.$$

(2) For any $\theta\in(0,1)$ there holds

$$\|j_f-j_g\|_{(C_b^{0,\theta}(\mathbb{R}^3))^*} \leq K_0\|j_f-j_g\|^{\frac{\theta}{\theta+1}}_{(C_b^{0,1}(\mathbb{R}^3))^*} \lesssim K_0\mathcal{M}_{k_0}^{\frac{1}{k_0}\frac{\theta}{(\theta+1)}}\, W_1(f,g)^{\frac{(k_0-1)}{k_0}\frac{\theta}{(\theta+1)}}.$$

Proof. These estimates are standard but we give the proof here for completeness.

(1) We shall first prove that

$$\|\rho_f-\rho_g\|_{(C_b^{0,\theta}(\mathbb{R}^3))^*} \lesssim W_1(\rho_f,\rho_g)^{\frac{\theta}{\theta+1}}, \tag{2.3}$$

from which we shall conclude by remarking that

$$W_1(\rho_f,\rho_g)\lesssim W_1(f,g). \tag{2.4}$$

Recall that

$$\|\rho_f-\rho_g\|_{(C_b^{0,\theta}(\mathbb{R}^3))^*} := \sup_{\|\phi\|_{C_b^{0,\theta}(\mathbb{R}^3)}\leq 1} \int_{\Omega_0} \phi(x)(\rho_f(\mathrm{d}x)-\rho_g(\mathrm{d}x)).$$

We consider a sequence of mollifiers $(\zeta_\epsilon)_{\epsilon\in(0,1)}$, that is, $\zeta_\epsilon(x)=\epsilon^{-3}\zeta(\epsilon^{-1}x)$, $\zeta\in C_c^\infty(\mathbb{R}^3)$ nonnegative, $\int\zeta(x)\,\mathrm{d}x=1$, and $\mathrm{supp}(\zeta)\subset B_1$. We split

$$\begin{aligned}\int_{\mathbb{R}^3}\phi(x)(\rho_f-\rho_g)(\mathrm{d}x) &= \int_{\mathbb{R}^3}(\phi*\zeta_\epsilon)(x)(\rho_f-\rho_g)(\mathrm{d}x) + \int_{\mathbb{R}^3}[\phi(x)-(\phi*\zeta_\epsilon)(x)](\rho_f-\rho_g)(\mathrm{d}x)\\ &=: T_1+T_2.\end{aligned}$$

For the term T_2, we easily remark that

$$\phi(x)-(\phi*\zeta_\epsilon)(x) = \int_{\mathbb{R}^3}[\phi(x)-\phi(x-y)]\zeta_\epsilon(y)\,\mathrm{d}y \leq [\phi]_{C^{0,\theta}}\int_{\mathbb{R}^3}|y|^\theta\zeta_\epsilon(y)\,\mathrm{d}y \leq [\phi]_{C^{0,\theta}}\,\epsilon^\theta,$$

where we introduced the classical notation for the semi-norm in $C^{0,\theta}$ on the right of these identities. Hence the previous estimate yields

$$T_2 \leq \|\phi-(\phi*\zeta_\epsilon)\|_{L^\infty}\int_{\mathbb{R}^3}(\rho_f+\rho_g)(\mathrm{d}x) \lesssim \|\phi\|_{C^{0,\theta}}\,\epsilon^\theta.$$

For the term T_1 we observe that, for $\epsilon > 0$ small enough, $x \mapsto (\phi * \zeta_\epsilon)(x)$ lies in $\mathrm{Lip}(\mathbb{R}^3)$, indeed, for any $x \in \mathbb{R}^3$, we have

$$\begin{aligned} |\nabla_x (\phi * \zeta_\epsilon)(x)| = |(\phi * \nabla_x \zeta_\epsilon)(x)| &\leq \int_{\mathbb{R}^3} |\phi(x-y)| \frac{|\nabla_x \zeta(y/\epsilon)|}{\epsilon^4} \, \mathrm{d}y \\ &\leq \int_{\mathbb{R}^3} |\phi(x-\epsilon w)| \frac{|\nabla \zeta(w)|}{\epsilon} \, \mathrm{d}w \\ &\lesssim \epsilon^{-1} \|\phi\|_{L^\infty} \|\nabla_x \zeta\|_{L^1}, \end{aligned}$$

which implies $[\phi * \zeta_\epsilon]_{\mathrm{Lip}} \lesssim \epsilon^{-1} \|\phi\|_{L^\infty}$. From that last estimate we get

$$\begin{aligned} T_1 &\lesssim [\phi * \zeta_\epsilon]_{\mathrm{Lip}} \int_{\mathbb{R}^3} \frac{\phi * \zeta_\epsilon(x)}{[\phi * \zeta_\epsilon]_{\mathrm{Lip}}} (\rho_f - \rho_g)(\mathrm{d}x) \\ &\lesssim \epsilon^{-1} \|\phi\|_{L^\infty} \sup_{[\psi]_{\mathrm{Lip}(\mathbb{R}^3)} \leq 1} \int_{\mathbb{R}^3} \psi(x)(\rho_f - \rho_g)(\mathrm{d}x) = \epsilon^{-1} \|\phi\|_{L^\infty} W_1(\rho_f, \rho_g). \end{aligned}$$

Gathering previous estimates and choosing $\epsilon = W_1(\rho_f, \rho_g)^{\frac{1}{\theta+1}}$ completes the proof of (2.3). We now easily prove (2.4) by remarking that

$$\begin{aligned} W_1(\rho_f, \rho_g) &= \sup_{[\psi]_{\mathrm{Lip}(\mathbb{R}^3)} \leq 1} \int_{\mathbb{R}^3} \psi(x)(\rho_f - \rho_g)(\mathrm{d}x) = \sup_{[\psi]_{\mathrm{Lip}(\mathbb{R}^3)} \leq 1} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \psi(x)(f-g)(\mathrm{d}x\mathrm{d}v) \\ &\leq \sup_{[\Psi]_{\mathrm{Lip}(\mathbb{R}^3 \times \mathbb{R}^3)} \leq 1} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \Psi(x,v)(f-g)(\mathrm{d}x, \mathrm{d}v) = W_1(f,g). \end{aligned}$$

(2) By reproducing *mutatis mutandis* the arguments for (2.3) we obtain

$$\|j_f - j_g\|_{(C^{0,\theta}_b(\mathbb{R}^3))^*} \lesssim \|j_f - j_g\|^{\frac{\theta}{\theta+1}}_{(C^{0,1}_b(\mathbb{R}^3))^*}. \tag{2.5}$$

So we prove next

$$\|j_f - j_g\|_{(C^{0,1}_b(\mathbb{R}^3))^*} \lesssim \mathcal{M}_k^{\frac{1}{k_0}} \, W_1(f,g)^{\frac{k_0 - 1}{k_0}}. \tag{2.6}$$

For $R > R_0$ (to be fixed later on) we define the smooth cutoff function $\chi_R(v) = \chi(|v|/R)$ with $\chi \in C^\infty_c(\mathbb{R})$ nonnegative and $\mathbf{1}_{B(0,1)} \leq \chi \leq \mathbf{1}_{B(0,2)}$, and we write

$$\begin{aligned} \|j_f - j_g\|_{(C^{0,1}_b(\mathbb{R}^3))^*} &:= \sup_{\|\phi\|_{C^{0,1}_b(\mathbb{R}^3)} \leq 1} \int_{\mathbb{R}^3} \phi(x)(j_f(\mathrm{d}x) - j_g(\mathrm{d}x)) = \sup_{\|\phi\|_{C^{0,1}_b(\mathbb{R}^3)} \leq 1} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \phi(x) v (f-g)(\mathrm{d}x, \mathrm{d}v) \\ &= \sup_{\|\phi\|_{C^{0,1}_b(\mathbb{R}^3)} \leq 1} \left\{ \int_{\mathbb{R}^3 \times \mathbb{R}^3} \phi(x) v \chi_R(v)(f-g)(\mathrm{d}x, \mathrm{d}v) + \int_{\mathbb{R}^3 \times \mathbb{R}^3} \phi(x) v (1 - \chi_R(v))(f-g)(\mathrm{d}x, \mathrm{d}v) \right\} \\ &=: I_1 + I_2. \end{aligned}$$

Observe that, given $\phi \in C^{0,1}_b(\mathbb{R}^3)$ such that $\|\phi\|_{C^{0,1}_b(\mathbb{R}^3)} \leq 1$, the mapping $(x,v) \mapsto \phi(x) v \chi_R(v)$ lies in $\mathrm{Lip}(\mathbb{R}^3 \times \mathbb{R}^3)$. Indeed, we have

$$\begin{aligned} &\sup_{(x,v) \neq (x',v')} \frac{|\phi(x) v \chi_R(v) - \phi(x') v' \chi_R(v')|}{(|x-x'|^2 + |v-v'|^2)^{1/2}} \\ &\quad \leq \sup_{(x,v) \neq (x',v')} \frac{|v \chi_R(v)| |\phi(x) - \phi(x')|}{(|x-x'|^2 + |v-v'|^2)^{1/2}} + \sup_{(x,v) \neq (x',v')} \frac{|\phi(x')| |\chi_R(v)| |v - v'|}{(|x-x'|^2 + |v-v'|^2)^{1/2}} \\ &\qquad + \sup_{(x,v) \neq (x',v')} \frac{|\phi(x')| |v'| \mathbf{1}_{|v'| \leq 2R} |\chi_R(v) - \chi_R(v')|}{(|x-x'|^2 + |v-v'|^2)^{1/2}} \\ &\quad \leq R \sup_{x \neq x'} \frac{|\phi(x) - \phi(x')|}{|x-x'|} + \sup_{v \neq v'} \frac{|\chi_R(v)| |v-v'|}{|v-v'|} + R \sup_{v \neq v'} \frac{|\chi_R(v) - \chi_R(v')|}{|v-v'|} \\ &\quad \lesssim R, \end{aligned}$$

which implies

$$I_1 \lesssim R W_1(f,g).$$

For the second term, since $f, g \in \mathcal{B}_{k_0}(C_2)$, we have

$$I_2 \lesssim \sup_{\|\phi\|_{C^{0,1}_b(\mathbb{R}^3)} \leq 1} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \phi(x) v (1 - \chi_R(v))(f-g)(\mathrm{d}x\mathrm{d}v) \lesssim \frac{\mathcal{M}_{k_0}}{R^{k_0 - 1}}$$

and we conclude to (2.6) by choosing $R = \frac{\mathcal{M}_{k_0}^{1/k_0}}{W_1(f,g)^{1/k_0}}$ if not infinite. □

With the above lemma we can show the following sufficient condition for the convergences in Assumption A2 to hold.

Lemma 2.3. *Consider a sequence $(\mathbf{Z}^N)_{N\in\mathbb{N}^*}$ of exchangeable random variables on $\mathcal{O}^N$ and the associated sequence of symmetric laws $(F^N)_{N\in\mathbb{N}^*}$ on $\mathcal{O}^N$ satisfying Assumption A1. Suppose that $(F^N)_{N\in\mathbb{N}^*}$ is f-chaotic (Definition 2.1), for some probability measure f on $\mathbb{R}^3\times\mathbb{R}^3$ with support on $\Omega_0\times\mathbb{R}^3$, and denote $\rho := \int_{\mathbb{R}^3} f(\cdot,\mathrm{d}v)$ and $j := \int_{\mathbb{R}^3} vf(\cdot,\mathrm{d}v)$. Then $(F^N)_{N\in\mathbb{N}^*}$ satisfies Assumption A2, more precisely there holds*

$$\mathbb{E}\left[W_1(\rho^N[\mathbf{Z}^N],\rho)\right] \lesssim \mathbb{E}\left[W_1(\mu^N[\mathbf{Z}^N],f)\right] \xrightarrow[N\to\infty]{} 0$$

and

$$\mathbb{E}\left[\|j^N[\mathbf{Z}^N]-j\|_{(C^{0,1}(\Omega))^*}\right] \lesssim \mathbb{E}\left[W_1(\mu^N[\mathbf{Z}^N],f)\right]^{\frac{k_0-1}{k_0}} \xrightarrow[N\to\infty]{} 0.$$

Proof. Thanks to the moment condition Assumption A1-(2) and the fact that $(F^N)_{N\in\mathbb{N}^*}$ is f-chaotic, we know from [13] that

$$\lim_{N\to\infty}\mathbb{E}\left[W_1(\mu^N[\mathbf{Z}^N],f)\right]=0.$$

We conclude the proof by applying Lemma 2.2 and remarking that $\mathbb{E}\left[M_{k_0}(\mu^N[\mathbf{Z}^N])\right]=M_{k_0}(F_1^N)$ is uniformly bounded thanks to Assumption A1-(2). □

2.2. **Estimating the weight of concentrated configurations.** For $\lambda,\alpha>0$, and any integer $M\le N$, we define

$$\mathcal{O}_\alpha^N := \left\{\mathbf{Z}^N\in\mathcal{O}^N \mid \min_{i\neq j}|X_i^N-X_j^N|<N^{-\alpha}\right\} \tag{2.7}$$

and

$$\mathcal{O}_{\lambda,M}^N := \left\{\mathbf{Z}^N\in\mathcal{O}^N \mid \text{there exist at least } M \text{ particles } (X_i^N) \text{ in the same cell } \mathcal{C}(\lambda) \text{ of size } \lambda>0\right\}. \tag{2.8}$$

Here the cell $\mathcal{C}(\lambda)$ is given by, for some $y\in\mathbb{R}^3$, $(y_1-\lambda/2,y_1+\lambda/2)\times(y_2-\lambda/2,y_2+\lambda/2)\times(y_3-\lambda/2,y_3+\lambda/2)$, so that $|\mathcal{C}_y(\lambda)|=\lambda^3$.

Below, we study the weight of the sets $\mathcal{O}_{\lambda,M}^N$ and $\mathcal{O}_\alpha$. For this, we allow that the parameters λ and M depend on N. Namely, we denote:

$$M_N = N^\beta, \qquad \lambda_N := \left(\eta\frac{M_N}{N}\right)^{1/3}, \quad \forall\, N\in\mathbb{N}^* \tag{2.9}$$

with positive parameters α,β,η to be fixed later on.

We now state the main result of this section.

Proposition 2.4. *Consider a sequence of random variables $(\mathbf{Z}^N)_{N\in\mathbb{N}^*}$ and the sequence of their associated laws $(F^N)_{N\in\mathbb{N}^*}$ satisfying Assumption A1. Let $\alpha\in(2/3,1)$, $\beta\in(0,1/2)$ and $\eta\in(0,\infty)$ sufficiently small. Then, the sequences $(M^N)_{N\in\mathbb{N}^*}$ and $(\lambda^N)_{N\in\mathbb{N}^*}$ given by formula (2.9) satisfy:*

$$\mathbb{P}(\mathbf{Z}^N\in\mathcal{O}_{\lambda_N,M_N}^N\cup\mathcal{O}_\alpha^N)\lesssim\frac{1}{N^{3\alpha-2}}\xrightarrow[N\to\infty]{}0.$$

We emphasize that the smallness restriction in the previous statement is explicit. With the notations of Assumption A1 it reads $\eta<1/(eC_1)$. The proof of Proposition 2.4 is split into the two following lemmas.

Lemma 2.5. *Under the assumptions of Proposition 2.4, there holds*

$$\mathbb{P}(\mathbf{Z}^N\in\mathcal{O}_{\lambda_N,M_N}^N)\lesssim(\eta C_1 e)^{N^\beta}.$$

Proof. By symmetry of F^N, given $\lambda>0$ and $M\in\mathbb{N}^*$ with $M\le N$, we have

$$\mathbb{P}(\mathbf{Z}^N\in\mathcal{O}_{\lambda,M}^N)=\binom{N}{M}\mathbb{P}((X_1^N,\dots,X_M^N)\text{ are in the same cell }\mathcal{C}(\lambda)).$$

In order to compute the last probability, again by symmetry, we only need to compute the probability of particles $i \in \{1, \dots, M-1\}$ to be in the same cell $\mathcal{C}(\lambda)$ containing X_M^N, the position of particle number M. Since a cell $\mathcal{C}(\lambda)$ has diameter λ (with respect to ℓ^∞-norm), we obtain:

$$\begin{aligned}
&\mathbb{P}(X_1^N, \dots, X_M^N \text{ are in the same cell } \mathcal{C}(\lambda)) \\
&\leq \mathbb{P}\left(\bigcap_{i=1}^{M-1} \{ |X_i^N - X_M^N|_\infty < \lambda \} \right) \\
&\leq \left(\lambda^3 C_1\right)^{M-1},
\end{aligned}$$

where we have used Assumption A1-(1) in last line.

When $N, M \to \infty$ with $(N-M) \to \infty$ and $N/M \to \infty$, Stirling's formula gives

$$\begin{aligned}
\binom{N}{M} = \frac{N!}{M!(N-M)!} &\sim \frac{\sqrt{2\pi} N^{1/2} N^N e^{-N}}{\sqrt{2\pi} M^{1/2} M^M e^{-M} \sqrt{2\pi} (N-M)^{1/2} (N-M)^{N-M} e^{-(N-M)}} \\
&\sim \frac{1}{\sqrt{2\pi}} \left(\frac{N}{M}\right)^M \frac{1}{M^{1/2}(1-\frac{M}{N})^{N-M+1/2}},
\end{aligned}$$

which implies

$$\mathbb{P}(\mathbf{Z}^N \in O_{\lambda,M}^N) \lesssim \frac{1}{\sqrt{2\pi}} \left(\frac{N}{M}\right)^M \frac{1}{M^{1/2}(1-\frac{M}{N})^{N-M+1/2}} \left(\lambda^3 C_1\right)^{M-1}.$$

We now consider the given sequences $(M_N)_{N\in\mathbb{N}^*}$ and $(\lambda_N)_{N\in\mathbb{N}^*}$ given by formula (2.9), and we get

$$\begin{aligned}
\mathbb{P}(\mathbf{Z}^N \in \mathcal{O}_{\lambda_N,M_N}^N) &\lesssim \left(\frac{N}{M_N}\right)^{M_N} \frac{1}{M_N^{1/2}(1-\frac{M_N}{N})^{N(1-M_N/N)+1/2}} \left(\eta \frac{M_N}{N} C_1\right)^{M_N-1} \\
&\lesssim \left(\frac{N}{M_N}\right) \frac{1}{M_N^{1/2}(1-\frac{M_N}{N})^{N(1-M_N/N)+1/2}} \left(\eta C_1\right)^{M_N-1}
\end{aligned}$$

Since $\beta \in (0, 1/2)$, we have that $M_N^2/N \to 0$ so that we can simplify the denominator of the right-hand side:

$$\begin{aligned}
\mathbb{P}(\mathbf{Z}^N \in \mathcal{O}_{\lambda_N,M_N}^N) &\lesssim \frac{N}{M_N^{3/2}} \left(\eta C_1 e\right)^{M_N-1} = N^{(1-\frac{3\beta}{2})} \left(\eta C_1 e\right)^{N^\beta - 1} \\
&\lesssim \exp\left(N^\beta \log(\eta C_1 e) + \left(1 - \tfrac{3\beta}{2}\right) \log N \right) \lesssim \left(\eta C_1 e\right)^{N^\beta}.
\end{aligned}$$

□

Lemma 2.6. *Under the assumptions of Proposition 2.4, there holds*

$$\mathbb{P}(\mathbf{Z}^N \in \mathcal{O}_\alpha^N) \lesssim C_1 N^{2-3\alpha}.$$

Proof. By symmetry of F^N we have

$$\mathbb{P}(\mathbf{Z}^N \in \mathcal{O}_\alpha^N) = \binom{N}{2} \mathbb{P}(|X_1^N - X_2^N| < N^{-\alpha}),$$

and we easily compute

$$\mathbb{P}(|X_1^N - X_2^N| < N^{-\alpha}) \lesssim \|F_1^N\|_{L_x^\infty L_v^1(\mathbb{R}^3\times\mathbb{R}^3)} N^{-3\alpha} \lesssim C_1 N^{-3\alpha},$$

which completes the proof. □

3. Properties of the mapping U_N for fixed N

In this section, we fix an arbitrary strictly positive $N \in \mathbb{N}$ and we analyze the properties of the mapping U^N. As N is fixed, we drop the exponents in notations (except $\mathcal{O}^N$). For example, we denote $U = U^N$, $\mathbf{X} = \mathbf{X}^N$, $\mathbf{V} = \mathbf{V}^N$, $X_i = X_i^N$ and $V_i = V_i^N$... The main result of this section reads:

Proposition 3.1. *The mapping U defined in* (1.4) *satisfies $U \in C(\mathcal{O}^N; \dot{H}^1(\mathbb{R}^3))$. Moreover, if $F \in L^1(\mathcal{O}^N)$ is a sufficiently regular symmetric probability density, we have $U \in L^1(\mathcal{O}^N, F(\mathbf{Z})\mathrm{d}\mathbf{Z})$.*

More quantitative statements on the integrability properties of U are stated in due course. In particular, the meaning of "F sufficiently regular" is made precise in Section 3.3 below.

Let first recall classical statement on the well-definition of the mapping U. For fixed $\mathbf{Z} \in \mathcal{O}^N$, by definition, the restriction u of $U[\mathbf{Z}]$ to

$$\mathcal{F} := \mathbb{R}^3 \setminus \bigcup_{i=1}^{N} \overline{B(X_i, 1/N)},$$

should be the unique $\dot{H}^1(\mathcal{F})$ vector-field for which there exists a pressure p such that (u,p) is a solution to:

$$\left\{ \begin{array}{rcl} -\Delta u + \nabla p & = & 0\,, \\ \operatorname{div} u & = & 0\,, \end{array} \right. \quad \text{in } \mathcal{F} \tag{3.1}$$

completed with boundary conditions:

$$\left\{ \begin{array}{rcl} u(x) & = & V_i \quad \text{on } B_i \\ \lim_{|x|\to\infty} u(x) & = & 0. \end{array} \right. \tag{3.2}$$

We recall here shortly the function spaces and analytical arguments underlying the mathematical treatment of this problem [14, Section 3]. We refer the interested reader also to [10, Sections IV-VI] for more details.

Given a smooth unbounded connected domain $\mathcal{F} \subset \mathbb{R}^3$ we denote :

$$\mathcal{D}_0(\mathcal{F}) := \{\varphi \in C^\infty_c(\mathcal{F}) \text{ s.t. } \operatorname{div}\varphi = 0\}\,, \qquad \mathcal{D}(\mathcal{F}) := \left\{\varphi_{|\mathcal{F}} \text{ with } \varphi \in C^\infty_c(\mathbb{R}^3) \text{ s.t. } \operatorname{div}\varphi = 0\right\}\,,$$

and $D(\mathcal{F})$ (resp. $D_0(\mathcal{F})$) the closure of $\mathcal{D}(\mathcal{F})$ (resp. $\mathcal{D}_0(\mathcal{F})$) for the $\dot{H}^1$-norm, namely:

$$\|u\|_{D(\mathcal{F})} = \left[\int_{\mathcal{F}} |\nabla u(x)|^2 \mathrm{d}x\right]^{\frac{1}{2}}.$$

We recall that (see [10, Theorem III.4.1 and Theorem III.4.2]):

$$D(\mathcal{F}) \quad = \quad \left\{ u \in L^2_{\mathrm{loc}}(\mathcal{F}) \text{ s.t. } \int_{\mathcal{F}} |\nabla u|^2 < \infty \text{ with } \operatorname{div} u = 0 \right\}\,,$$

and that $D(\mathcal{F})$ and $D_0(\mathcal{F})$ are Hilbert spaces when endowed with the scalar product

$$(u,v) \mapsto \int_{\mathcal{F}} \nabla u : \nabla v.$$

We recall also that $D(\mathcal{F}) \subset L^6(\mathcal{F})$. In particular, since $\partial\mathcal{F}$ is compact, there exists a linear continuous trace operator $\gamma_0 : D(\mathcal{F}) \to H^{\frac{1}{2}}(\partial\mathcal{F})$ such that $D_0(\mathcal{F}) = \operatorname{Ker}\gamma_0$.

With these definitions, problem (3.1)-(3.2) is associated with a(n equivalent) weak formulation:

Find $u \in D(\mathcal{F})$ such that $\gamma_0(u) = V_i$ on ∂B_i for $i = 1,\dots,N$ and

$$\int_{\mathcal{F}} \nabla u : \nabla w = 0\,, \quad \text{for arbitrary } w \in D_0(\mathcal{F}).$$

Existence of a weak-solution yields by applying a standard Riesz-Fréchet or Lax-Milgram argument which also yields the following variational property:

Theorem 3.2. *The vector-field $U[\mathbf{Z}] \in D(\mathbb{R}^3)$ is the unique minimizer of*

$$\left\{ \int_{\mathbb{R}^3} |\nabla v|^2, \quad v \in D(\mathbb{R}^3) \text{ s.t. } v_{|B_i} = V_i \quad \text{for all } i \in \{1,\dots,N\} \right\}.$$

We refer the reader to [14, Theorem 3] for a proof. The remainder of this section is organized as follows. In the next subsection, we consider the continuity properties of the mapping U. We continue by deriving a pointwise estimate and end up the section with an analysis of integrability properties of U.

3.1. **Continuity of the mapping U.** At first, we obtain that:

Lemma 3.3. *The mapping U satisfies $U \in C(\mathcal{O}^N; D(\mathbb{R}^3))$.*

As only continuity is required for our purpose, we give below a proof of this lemma based on monotonicity arguments only. Nonetheless, one may prove much finer properties by using change of variables methods (see [21, 4] for instance).

Proof. The problem (3.1)-(3.2) being linear with respect to its boundary data, we have that, for fixed $\mathbf{X} \in \mathbb{R}^{3N}$ such that $|X_i - X_j| > 2/N$ when $i \neq j$, the mapping $\mathbf{V} \mapsto U[\mathbf{Z}]$ is linear. Consequently, it is sufficient to consider the continuity of the mapping $\mathbf{X} \mapsto U[\mathbf{Z}]$ for fixed $\mathbf{V}$.

Let $\mathbf{V} \in \mathbb{R}^{3N}$ be fixed and consider $\mathbf{X} \in \mathbb{R}^{3N}$ – such that $|X_i - X_j| > 2/N$ for any $i \neq j$ – and a sequence $(\mathbf{X}^{(k)})_{k\in\mathbb{N}}$ in $\mathbb{R}^{3N}$ such that

- $\mathbf{Z}^{(k)} = (\mathbf{X}^{(k)}, \mathbf{V}) \in \mathcal{O}^N$ for any $k \in \mathbb{N}$,
- $\lim_{k\to\infty} X_i^{(k)} = X_i$, for $i = 1, \dots, N$.

We are interested in proving that $U[\mathbf{Z}^{(k)}]$ converges to $U[\mathbf{Z}]$ in $D(\mathbb{R}^3)$. Due to the variational characterization of $U[\mathbf{Z}]$, we remark that it is sufficient to prove that the sequence $(m^{(k)})_{k\in\mathbb{N}}$ defined by

$$m^{(k)} := \inf \left\{ \int_{\mathbb{R}^3} |\nabla v|^2, \quad v \in D(\mathbb{R}^3) \text{ s.t. } v_{|_{B(X_i^{(k)},1/N)}} = V_i \quad \text{for all } i \in \{1, \dots, N\} \right\} \quad \forall\, k \in \mathbb{N}$$

satisfies:

$$\lim_{k\to\infty} m^{(k)} = m_\infty := \inf \left\{ \int_{\mathbb{R}^3} |\nabla v|^2, \quad v \in D(\mathbb{R}^3) \text{ s.t. } v_{|_{B(X_i,1/N)}} = V_i \quad \text{for all } i \in \{1, \dots, N\} \right\}. \tag{3.3}$$

Indeed, for arbitrary $k \in \mathbb{N}$, there holds: $m^{(k)} = \|\nabla U[\mathbf{Z}^{(k)}]\|^2_{L^2(\mathbb{R}^3)}$. Consequently, if $(m^{(k)})_{k\in\mathbb{N}}$ converges, $U[\mathbf{Z}^{(k)}]$ is bounded in $D(\mathbb{R}^3)$. We may then pass to the limit in the weak formulation of the Stokes problem (restricted to test-function in $\mathcal{D}_0(\mathcal{F})$) and we obtain that $U[\mathbf{Z}]$ is the weak limit of $U[\mathbf{Z}^{(k)}]$ in $D(\mathbb{R}^3)$. The convergence of $(m^{(k)})_{k\in\mathbb{N}}$ implies then that $(\|\nabla U[\mathbf{Z}^{(k)}]\|_{L^2(\mathbb{R}^3)})_{k\in\mathbb{N}}$ converges to $\|\nabla U[\mathbf{Z}]\|_{L^2(\mathbb{R}^3)}$. As $D(\mathbb{R}^3)$ is a Hilbert space, this ends the proof.

To prove (3.3), we analyze the continuity properties of the function $m_\infty(\cdot)$ as defined by:

$$m_\infty(R) = \inf \left\{ \int_{\mathbb{R}^3} |\nabla v|^2, \quad v \in D(\mathbb{R}^3) \text{ s.t. } v_{|_{B(X_i,R)}} = V_i \quad \text{for all } i \in \{1, \dots, N\} \right\}, \quad \forall\, R > 0,$$

We note that $m_\infty = m_\infty(1/N)$ and that, as $|X_i - X_j| > 2/N$ for $i \neq j$, this function is well defined for R close to $1/N$. Left continuity in $1/N$ is for free. Indeed, by construction, $m_\infty(\cdot)$ is increasing and, if we had $\lim_{R\to 1/N^-} m_\infty(R) < m_\infty(1/N)$, we would be able to construct a vector-field $v \in D(\mathbb{R}^3)$ satisfying simultaneously $v_{|_{B(X_i,1/N)}} = V_i$ for $i = 1, \dots, N$ and

$$\int_{\mathbb{R}^3} |\nabla v|^2 \le \lim_{R\to 1/N^-} m_\infty(R) < m_\infty(1/N),$$

which yields a contradiction. Right continuity in $1/N$ is a bit more intricate. To this end, we note that $m_\infty(1/N)$ is achieved by $U[\mathbf{Z}]$. Remarking that, on the one hand, for an arbitrary truncation function χ there holds:

$$\nabla \times [\chi(x) V_i \times x] = \begin{cases} V_i & \text{on the set } \{\chi = 1\} \\ 0 & \text{on the set } \{\chi = 0\}, \end{cases}$$

and that, on the other hand $\mathcal{D}_0(\mathcal{F}^N)$ is dense in $D_0(\mathcal{F}^N)$, we may construct a sequence $(w^{(l)})_{l\in\mathbb{N}} \in [D(\mathbb{R}^3)]^{\mathbb{N}}$ converging to $U[\mathbf{Z}]$ and a sequence $(\varepsilon^{(l)})_{l\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$ converging to 0 such that, for arbitrary l there holds:

$$w^{(l)} = V_i \quad \text{on } B(X_i, 1/N + \varepsilon^{(l)})\,, \quad \forall\, i = 1, \dots, N.$$

This implies that:

$$\|\nabla U[\mathbf{Z}]\|_{L^2(\mathbb{R}^3)} = m_\infty(1/N) \le m_\infty(1/N + \varepsilon^{(l)}) \le \|\nabla w^{(l)}\|_{L^2(\mathbb{R}^3)}\,, \quad \forall\, k \in \mathbb{N},$$

and consequently, by comparison, that:

$$\lim_{R\to 1/N^+} m_\infty(R) = \lim_{l\to\infty} m_\infty(1/N + \varepsilon^{(l)}) = m_\infty(1/N).$$

To conclude, we apply a simple geometric argument implying that, associated with the sequence $(X^{(k)})_{k\in\mathbb{N}}$, we may construct a sequence $(\eta^{(k)})_{k\in\mathbb{N}} \in (0,\infty)$ converging to 0 for which, for arbitrary $k \in \mathbb{N}$ we have:

$$B(X_i, 1/N - \eta^{(k)}) \subset B(X_i^{(k)}, 1/N) \subset B(X_i, 1/N + \eta^{(k)}) \quad \forall\, i = 1, \dots, N.$$

Consequently, for arbitrary $k \in \mathbb{N}$, by comparing the sets on which $U[\mathbf{Z}^{(k)}]$ is equal to V_i with balls of center X_i, we obtain:

$$m_\infty(1/N - \eta^{(k)}) \le m^{(k)} \le m_\infty(1/N + \eta^{(k)}).$$

We conclude the proof thanks to the previous continuity analysis of $R \mapsto m_\infty(R)$ in $R = 1/N$. $\square$

3.2. **A pointwise estimate.** We obtain now a bound for given configurations:

Lemma 3.4. *There exists a universal constant C for which, given $\mathbf{Z} \in \mathcal{O}^N$, there holds:*

$$\|\nabla U[\mathbf{Z}]\|^2_{L^2(\mathbb{R}^3)} \leq \frac{C}{N}\sum_{i=1}^N |V_i|^2 \left(1 + \sum_{j\neq i} \frac{\mathbf{1}_{|X_i - X_j| < \frac{5}{2N}}}{|X_i - X_j| - \dfrac{2}{N}}\right).$$

Proof. In this proof $\mathbf{Z} \in \mathcal{O}^N$ is fixed and splits into $\mathbf{X}$ and $\mathbf{V}$. The idea of the proof is to construct a suitable function

$$w \in Y[\mathbf{Z}] := \Big\{ v \in D(\mathbb{R}^3) \text{ s.t. } v_{|_{B_i}} = V_i \text{ for all } i \in \{1, \dots, N\}\Big\}$$

whose norm can be bounded by the right-hand side of the above inequality. The bound is then transferred to $U[\mathbf{Z}]$ *via* its variational characterization (see **Theorem 3.2**). Technical details are rather long, hence we stick to the main ideas here and postpone them to the appendix.

To construct a candidate w, we treat all the B_i independently. In Appendix A, we prove:

Lemma 3.5. *Given $i \in \{1, \dots, N\}$, there exists $w_i \in D(\mathbb{R}^3)$ satisfying*

$$w_i = V_i \text{ in } B_i \text{ and } w_i = 0 \text{ in } B_j \text{ for } j \neq i\,, \tag{3.4}$$

$$\mathrm{Supp}(w_i) \subset B(X_i, \tfrac{3}{2N}), \tag{3.5}$$

such that:

$$\|\nabla w_i\|^2_{L^2(\mathbb{R}^3)} \leq \frac{C|V_i|^2}{N} \left(1 + \sum_{j\neq i} \frac{\mathbf{1}_{|X_i - X_j| < \frac{5}{2N}}}{|X_i - X_j| - \dfrac{2}{N}}\right). \tag{3.6}$$

for a universal constant C.

Let $(w_i)_{i=1,\dots,N}$ be given by Lemma 3.5. By combining (3.4) for $i = 1, \dots, N$, it is straightforward that:

$$w = \sum_{i=1}^N w_i \in Y[\mathbf{Z}].$$

Furthermore:

$$\int_{\mathbb{R}^3} |\nabla w|^2 = \sum_{i=1}^N \sum_{j=1}^N \int_{\mathbb{R}^3} \nabla w_i : \nabla w_j.$$

At this point, we use the property (3.5) in order to bound the second term on the right-hand side. Given $i \in \{1, \dots, N\}$ let denote

$$\mathcal{I}_i := \left\{ j \in \{1, \dots, N\} \text{ s.t. } B(X_i, \tfrac{3}{2N}) \cap B(X_j, \tfrac{3}{2N}) \neq \emptyset \right\}.$$

We remark that, given two indices i and j we have the equivalence between $j \in \mathcal{I}_i$ and $i \in \mathcal{I}_j$.

On the one hand, applying (3.5), there holds:

$$\sum_{j=1}^N \int_{\mathbb{R}^3} \nabla w_i : \nabla w_j = \sum_{j \in \mathcal{I}_i} \int_{\mathbb{R}^3} \nabla w_i : \nabla w_j \quad \forall\, i = 1, \dots, N.$$

On the other hand, we have:

Lemma 3.6. *Given $i \in \{1, \dots, N\}$ the set $\mathcal{I}_i$ contains at most 16 distinct indices.*

This lemma is obtained thanks to simple geometric argument that we develop in Appendix A. Applying standard inequalities, we can then bound:

$$\left|\sum_{j=1}^N \int_{\mathbb{R}^3} \nabla w_i : \nabla w_j\right| \leq 8 \int_{\mathbb{R}^3} |\nabla w_i|^2 + \frac{1}{2}\sum_{j \in \mathcal{I}_i} \int_{\mathbb{R}^3} |\nabla w_j|^2, \quad \forall\, i = 1, \dots, N,$$

which entails:

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla w|^2 &\leq 8 \sum_{i=1}^N \int_{\mathbb{R}^3} |\nabla w_i|^2 + \frac{1}{2} \sum_{i=1}^N \sum_{j \in \mathcal{I}_i} \int_{\mathbb{R}^3} |\nabla w_j|^2 \\ &\leq 8 \sum_{i=1}^N \int_{\mathbb{R}^3} |\nabla w_i|^2 + \frac{1}{2} \sum_{j=1}^N |\mathcal{I}_j| \int_{\mathbb{R}^3} |\nabla w_j|^2. \\ &\leq 16 \sum_{i=1}^N \int_{\mathbb{R}^3} |\nabla w_i|^2.\end{aligned}$$

We then conclude the proof by applying (3.6). □

3.3. **Integrability properties of the mapping U.** In this last part, we envisage to integrate the mapping U against a sufficiently regular symmetric probability density $F \in L^1(\mathcal{O}^N)$. To state the regularity assumption, we recall the notations:

$$\begin{aligned} F_1(z) &= \int_{\mathbb{R}^{6(N-1)}} \mathbf{1}_{\mathcal{O}^N}(z, \mathbf{z}') F(z, \mathbf{z}') \mathrm{d}\mathbf{z}', \quad \forall\, z \in \mathbb{R}^6, \\ F_2(z_1, z_2) &= \int_{\mathbb{R}^{6(N-2)}} \mathbf{1}_{\mathcal{O}^N}(z_1, z_2, \mathbf{z}') F(z_1, z_2, \mathbf{z}') \mathrm{d}\mathbf{z}', \quad \forall\, (z_1, z_2) \in \mathcal{O}_2^N, \end{aligned}$$

where $\mathcal{O}_2^N := \left\{ (z_1, z_2) \in \mathbb{R}^6 \text{ s.t. } |x_1 - x_2| > \dfrac{2}{N} \right\}$. We introduce also:

$$\mathfrak{j}(x_1, x_2) = \int_{\mathbb{R}^6} |v_1| F_2((x_1, v_1), (x_2, v_2)) \mathrm{d}v_1 \mathrm{d}v_2, \quad \forall\, (x_1, x_2) \text{ s.t. } |x_2 - x_1| > \frac{2}{N}.$$

With these notations, we prove

Proposition 3.7. *Let $F \in L^1(\mathcal{O}^N)$ be a symmetric probability density satisfying*

$$\int_{\mathbb{R}^6} (1 + |z|^2) F_1(z) \mathrm{d}z < \infty, \tag{3.7}$$

$$\int_{\mathbb{R}^3} \left[\sup_{x_2 \in \mathbb{R}^3 \setminus \overline{B(x_1, 2/N)}} |\mathfrak{j}(x_1, x_2)| \right] \mathrm{d}x_1 < \infty. \tag{3.8}$$

There holds $U \in L^1(\mathcal{O}^N, F(\mathbf{Z})\mathrm{d}\mathbf{Z})$ and there exists a universal constant C such that:

$$\mathbb{E}[\|\nabla U\|_{L^2(\mathbb{R}^3)}] \leq C \left[\left(\int_{\mathbb{R}^6} (1 + |z|^2) F_1(z) \mathrm{d}z \right)^{\frac{1}{2}} + \frac{1}{N} \int_{\mathbb{R}^3} \left[\sup_{x_2 \in \mathbb{R}^3 \setminus \overline{B(x_1, 2/N)}} |\mathfrak{j}(x_1, x_2)| \right] \mathrm{d}x_1 \right].$$

Proof. Let $\mathbf{Z} \in \mathcal{O}^N$, applying the bound of Lemma 3.4 together with a standard comparison argument, we obtain that:

$$\|\nabla U[\mathbf{Z}]\|_{L^2(\mathbb{R}^3)} \leq C \left[\frac{1}{\sqrt{N}} \left[\sum_{i=1}^N |V_i|^2 \right]^{\frac{1}{2}} + \frac{1}{\sqrt{N}} \sum_{i=1}^N \sum_{j \neq i} |V_i| \frac{\mathbf{1}_{|X_i - X_j| < \frac{5}{2N}}}{\sqrt{|X_i - X_j| - \frac{2}{N}}} \right].$$

We have then

$$\mathbb{E}[\|\nabla U[\mathbf{Z}]\|_{L^2(\mathbb{R}^3)}] \leq C \left(\mathbb{E}\left[\left(\frac{1}{N} \sum_{i=1}^N |V_i^N|^2 \right)^{\frac{1}{2}} \right] + \mathbb{E}\left[\frac{1}{\sqrt{N}} \sum_{i=1}^N \sum_{j \neq i} |V_i| \frac{\mathbf{1}_{|X_i - X_j| < \frac{5}{2N}}}{\sqrt{|X_i - X_j| - \frac{2}{N}}} \right] \right)$$

We split the right-hand side into two integrals I_1 and I_2. First applying a Jensen inequality and then symmetry properties of the measure F we have:

$$\begin{aligned} I_1 &:= \mathbb{E}\left[\left(\frac{1}{N} \sum_{i=1}^N |V_i^N|^2 \right)^{\frac{1}{2}} \right] \leq \mathbb{E}\left[\frac{1}{N} \sum_{i=1}^N |V_i^N|^2 \right]^{\frac{1}{2}} \\ &\leq \left(\int_{\mathbb{R}^6} (1 + |z|^2) F_1(z) \mathrm{d}z \right)^{1/2}. \end{aligned}$$

Furthermore, using symmetry,

$$
\begin{aligned}
I_2 &:= \mathbb{E}\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\sum_{j\neq i}|V_i|\,\frac{\mathbf{1}_{|X_i-X_j|<\frac{5}{2N}}}{\sqrt{|X_i-X_j|-\frac{2}{N}}}\right]\\
&\leq N^{\frac{3}{2}}\mathbb{E}\left[|V_1|\,\frac{\mathbf{1}_{|X_1-X_2|<\frac{5}{2N}}}{\sqrt{|X_1-X_2|-\frac{2}{N}}}\right]\\
&= N^{\frac{3}{2}}\int_{|x_1-x_2|>\frac{2}{N}}\left\{\int_{\mathbb{R}^3\times\mathbb{R}^3}|v_1|\frac{\mathbf{1}_{|x_1-x_2|<\frac{5}{2N}}}{\sqrt{|x_1-x_2|-\frac{2}{N}}}F_2(z_1,z_2)\mathrm{d}v_1\mathrm{d}v_2\right\}\mathrm{d}x_1\mathrm{d}x_2\\
&\leq N^{\frac{3}{2}}\int_{\mathbb{R}^3}\int_{B(x_1,\frac{5}{2N})\setminus B(x_1,\frac{2}{N})}\frac{1}{\sqrt{|x_1-x_2|-\frac{2}{N}}}\,\mathrm{j}(x_1,x_2)\,\mathrm{d}x_2\mathrm{d}x_1.\\
&\leq \frac{1}{N}\int_{\mathbb{R}^3}\left[\sup_{x_2\in\mathbb{R}^3\setminus\overline{B(x_1,2/N)}}|\mathrm{j}(x_1,x_2)|\mathrm{d}x_1\right]\int_{B(0,\frac{5}{2})\setminus B(0,2)}\frac{1}{\sqrt{|y|-2}}\,\mathrm{d}y.
\end{aligned}
$$

This ends the proof. □

With similar arguments as in the proof of this theorem, we also obtain the following corollary:

Corollary 3.8. *Under the assumptions of Proposition 3.7, given $\tilde{\mathcal{O}}^N\subset\mathcal{O}^N$ we have:*

$$
\mathbb{E}[\|\nabla U\|_{L^2(\mathbb{R}^3)}\mathbf{1}_{\tilde{\mathcal{O}}_N}]\leq C\left[|\mathbb{P}(\tilde{\mathcal{O}}_N)|^{\frac{1}{2}}\left(\int_{\mathbb{R}^6}(1+|z|^2)F_1(z)\mathrm{d}z\right)^{\frac{1}{2}}+\frac{1}{N}\int_{\mathbb{R}^3}\left[\sup_{x_2\in\mathbb{R}^3\setminus\overline{B(x_1,2/N)}}|\mathrm{j}(x_1,x_2)|\right]\mathrm{d}x_1\right].
$$

4. Main estimate for non-concentrated configurations

In this section, we compute a quantitative bound for the distance between a solution to the N-particle problem and the limit Stokes-Brinkman system in a "favorable" case. For this, let first state a stability estimate for the Stokes-Brinkman system suitable to our purpose.

Let consider a nonnegative density $\tilde{\rho}\in L^3(\Omega_0)$ and a momentum $\tilde{\jmath}\in L^2(\mathcal{O})$ where Ω_0 and $\mathcal{O}$ are bounded open subsets of $\mathbb{R}^3$. The subset Ω_0 is the one given in the introduction, corresponding to the domain occupied by the cloud of particles. We denote below $\Omega_1=\Omega_0+B(0,1)$. The subset $\mathcal{O}$ is another bounded open subset, not necessarily the same one. We apply the convention that we extend $\tilde{\rho}$ and $\tilde{\jmath}$ by 0 in order to yield functions on $\mathbb{R}^3$. In this framework, the existence/uniqueness theorem in bounded domains (as mentioned in [19]) extends to the Stokes-Brinkman problem on the whole space:

$$
\left\{\begin{array}{rcl}-\Delta u+\nabla p+6\pi\tilde{\rho}u &=& 6\pi\tilde{\jmath}\\ \operatorname{div}u &=& 0\end{array}\right.\qquad\text{in }\mathbb{R}^3,\tag{4.1}
$$

$$
\lim_{|x|\to\infty}|u(x)|=0.\tag{4.2}
$$

Indeed, as in the case of the Stokes problem, the system (4.1)-(4.2) is associated with the weak formulation

Find $u\in D(\mathbb{R}^3)$ such that

$$
\int_{\mathbb{R}^3}\nabla u:\nabla w+6\pi\int_{\mathbb{R}^3}\tilde{\rho}u\cdot w=6\pi\int_{\mathbb{R}^3}\tilde{\jmath}\cdot w,\qquad\forall\,w\in D(\mathbb{R}^3).
$$

For positive $\tilde{\rho}\in L^3(\Omega_0)\subset L^{3/2}(\mathbb{R}^3)$, the left-hand side of the weak formulation represents a bilinear mapping a_ρ which is in the same time coercive and continuous on $D(\mathbb{R}^3)$ (we recall that $D(\mathbb{R}^3)\subset L^6(\mathbb{R}^3)$). Hence, for arbitrary $\tilde{\jmath}\in L^2(\Omega_0)\subset L^{6/5}(\mathbb{R}^3)\subset[D(\mathbb{R}^3)]^*$ we can apply a standard Lax-Milgram argument to obtain that (4.1)-(4.2) admits a unique weak solution $u:=u[\tilde{\rho},\tilde{\jmath}]\in D(\mathbb{R}^3)$. At this point, we note that any weak solution u to (4.1)-(4.2) is also a weak solution to the Stokes equations with data $6\pi(\tilde{\jmath}-\tilde{\rho}u)$. Since $\tilde{\jmath}\in L^2(\mathbb{R}^3)$ and $\tilde{\rho}\in L^3(\mathbb{R}^3)$ we obtain that the source term is in $L^2(\mathbb{R}^3)$ and apply elliptic regularity estimates for the Stokes equations on $\mathbb{R}^3$ (see [10, Theorem IV.2.1]). This yields:

Proposition 4.1. *For arbitrary $\tilde{\jmath} \in L^2(\mathcal{O})$ and non-negative $\tilde\rho \in L^3(\Omega_0)$ the unique weak solution $u := u[\tilde\rho,\tilde{\jmath}]$ to the Stokes-Brinkman problem (4.1)-(4.2) satisfies $\nabla^2 u \in L^2(\mathbb{R}^3)$ and there exists constants K_0, K_1 whose dependencies are mentioned in parenthesis such that:*

$$\|\nabla u\|_{L^2(\mathbb{R}^3)} \leq K_0 \|\tilde{\jmath}\|_{L^{6/5}(\mathbb{R}^3)}, \qquad \|\nabla^2 u\|_{L^2(\mathbb{R}^3)} \leq K_1(\|\tilde\rho\|_{L^3(\mathbb{R}^3)}) \left[\|\tilde{\jmath}\|_{L^2(\mathbb{R}^3)} + \|\tilde{\jmath}\|_{L^{6/5}(\mathbb{R}^3)}\right].$$

By duality, the previous elliptic-regularity statement entails a regularity statement in negative Sobolev spaces. Namely, given a nonnegative density $\tilde\rho \in L^3(\Omega_0)$, we denote, for arbitrary $v \in D(\mathbb{R}^3)$:

$$[v]_{\tilde\rho,2} := \sup\left\{ \left| \int_{\mathbb{R}^3} \nabla v : \nabla w + 6\pi \int_{\mathbb{R}^3} \tilde\rho v \cdot w \right|, w \in \mathcal{D}(\mathbb{R}^3) \text{ with } \|\nabla w\|_{L^2(\mathbb{R}^3)} + \|\nabla^2 w\|_{L^2(\mathbb{R}^3)} \leq 1 \right\}.$$

Reproducing the arguments of [19, Lemma 2.4], we obtain then the following proposition:

Proposition 4.2. *Given a bounded open subset $\mathcal{O} \subset \mathbb{R}^3$, there exists $K := K(\mathcal{O}, \|\tilde\rho\|_{L^3(\Omega_0)})$ such that*

$$\|v\|_{L^2(\mathcal{O})} \leq K[v]_{2,\mathbb{R}^3}.$$

We refer the reader to the proof of [19, Lemma 2.4] for more details.

Below, we apply this latter proposition to compare $U_N[\mathbf{Z}^N]$ for a given configuration $\mathbf{Z}^N$ with $u[\rho, j]$ where ρ, j are the density/flux associated with the distribution f given by the assumptions in Theorem 1.1. For this, we interpret $U_N[\mathbf{Z}^N]$ as the solution to a Stokes-Brinkman problem on $\mathbb{R}^3$ with measure data. We emphasize that a standard estimate in $\dot{H}^1$ is too greedy in terms of the regularity of the data so that we have to turn to the stability estimate given by Proposition 4.1 for our purpose.

4.1. **Main result of this section.** To state the main result of this section, we recall the notations introduced in [14] to handle the convergence of U_N towards $u[\rho, j]$. Given $N \in \mathbb{N}^*$ and $\mathbf{Z} = (X_1, V_1, \ldots, X_N, V_N) \in \mathcal{O}^N$, we denote:

- $d_{\min}[\mathbf{Z}]$ the minimal distance between two different centers X_i;
- $\lambda[\mathbf{Z}]$ a chosen size for a partition of $\mathbb{R}^3$ in cubes;
- $M[\mathbf{Z}]$ the maximum number of centers X_i inside one cell of size $\lambda[\mathbf{Z}]$.

If $d_{\min}[\mathbf{Z}]$ is sufficiently large and $M[\mathbf{Z}]$ is sufficiently small, the particles are distant and do not concentrate in a small box. This is the reason for the name "non-concentrated configurations" of this section. With these latter notations, the main result of this section is the following estimate:

Theorem 4.3. *Let $\alpha \in (2/3, 1)$, $\eta \in (0,1)$, $R > 0$ and $\delta > 1/2$ be given. There exists a positive constant $K := K(\alpha, R, \Omega_0)$ such that, for $N \geq 1$, given $\mathbf{Z}^N \in \mathcal{O}^N$ such that*

$$d_{\min}[\mathbf{Z}^N] \geq \frac{1}{N^\alpha}, \quad M[\mathbf{Z}^N] \leq \frac{N^{3(1-\alpha)/5}}{\eta}, \quad \lambda[\mathbf{Z}^N] = \left(\frac{\eta M[\mathbf{Z}^N]}{N}\right)^{\frac{1}{3}}, \tag{4.3}$$

we have

$$\|U_N[\mathbf{Z}^N] - u[\rho, j]\|_{L^2(B(0,R))} \leq \frac{K}{\eta}\Bigg[\|j[\mathbf{Z}^N] - j\|_{[C_b^{0,1/2}(\mathbb{R}^3)]^*} + \left(1 + \frac{1}{N}\sum_{i=1}^N |V_i^N|^2\right)^{\frac{5}{4}} \left(\frac{1 + \|\rho\|_{L^2(\Omega_0)}}{\sqrt{\delta}} + \delta^6 \left(\frac{1}{N^{\frac{1-\alpha}{5}}} + \|\rho[\mathbf{Z}^N] - \rho\|_{[C_b^{0,1/2}(\mathbb{R}^3)]^*} \right) \right) \Bigg].$$

where we recall that

$$\rho[\mathbf{Z}^N] = \frac{1}{N}\sum_{i=1}^N \delta_{X_i^N}, \qquad j[\mathbf{Z}^N] = \frac{1}{N}\sum_{i=1}^N V_i^N \delta_{X_i^N}.$$

The remainder of this section is devoted to the proof of this theorem. It is based on interpolating the method of [19] for dilute suspensions with the construction of [14]. Though the computations follow the line of these previous reference, we give an extensive proof for completeness because estimates have to be adapted at each line.

Proof of Theorem 4.3. From now on, we pick α, η, δ, R as in the assumptions of our Theorem 4.3, $N \geq 1$ and $\mathbf{Z} = (X_1, V_1, \ldots, X_N, V_N) \in \mathcal{O}^N$ such that (4.3) hold true. For legibility, we forget the N-dependencies in many notations in the proof. We recall that, by assumption, $\mathrm{Supp}(\rho[\mathbf{Z}]) \cup \mathrm{Supp}(j[\mathbf{Z}]) \subset \Omega_0$ and we denote $\Omega_1 := \Omega_0 + B(0,1)$.

To begin with, we note that, by applying the variational characterization associated with the Stokes problem (see [14, Theorem 3]), we can construct a constant C_0 such that:

$$\|\nabla U[\mathbf{Z}]\|^2_{L^2(\mathbb{R}^3)} \leq \frac{C_0}{N} \sum_{i=1}^N |V_i|^2. \tag{4.4}$$

This property relies mostly on the fact that $N d_{\min}[\mathbf{Z}]$ is bounded below by a strictly positive constant. We refer the reader to [14, Section 3] for more details.

We want to compute a bound by above on $\|U[\mathbf{Z}] - u[\rho, j]\|_{L^2(B(0,R))}$. Applying **Proposition 4.2**, this reduces to compute a bound for:

$$[U[\mathbf{Z}] - u[\rho,j]]_{2,\mathbb{R}^3} := \sup\Big\{ \Big| \int_{\mathbb{R}^3} \nabla(U[\mathbf{Z}] - u[\rho,j]) : \nabla w + 6\pi \int_{\mathbb{R}^3} \rho(U[\mathbf{Z}] - u[\rho,j]) \cdot w \Big| , w \in \mathcal{D}(\mathbb{R}^3) \text{ with } \|\nabla w\|_{L^2(\mathbb{R}^3)} + \|\nabla^2 w\|_{L^2(\mathbb{R}^3)} \leq 1 \Big\},$$

or to find a constant K independent of $U[\mathbf{Z}]$ and $w \in \mathcal{D}(\mathbb{R}^3)$ for which there holds

$$\left| \int_{\mathbb{R}^3} \nabla(U[\mathbf{Z}] - u[\rho,j]) : \nabla w + 6\pi \int_{\mathbb{R}^3} \rho(U[\mathbf{Z}] - u[\rho,j]) \cdot w \right| \leq K \left[\|\nabla w\|_{L^2(\mathbb{R}^3)} + \|\nabla^2 w\|_{L^2(\mathbb{R}^3)} \right].$$

Hence, in what follows we fix $w \in \mathcal{D}(\mathbb{R}^3)$ and we focus on:

$$E[w] := \int_{\mathbb{R}^3} \nabla[U[\mathbf{Z}] - u[\rho,j]] : \nabla w + 6\pi \int_{\mathbb{R}^3} \rho(U[\mathbf{Z}] - u[\rho,j]) \cdot w.$$

We apply without mention below that, since Ω_1 is bounded, there holds:

$$\|w\|_{C^{0,1/2}(\overline{\Omega_1})} + \|\nabla w\|_{L^6(\Omega_1)} \lesssim \|\nabla w\|_{L^2(\mathbb{R}^3)} + \|\nabla^2 w\|_{L^2(\mathbb{R}^3)} =: \|w\|_{D^2(\mathbb{R}^3)}.$$

First, we decompose the error term $E[w]$ into several pieces that are treated independently in the rest of the proof. Since $u[\rho,j]$ is the weak solution to the Stokes-Brinkman problem associated with (ρ, j), this error term rewrites:

$$\begin{aligned} E[w] &= \int_{\mathbb{R}^3} (\nabla U[\mathbf{Z}] : \nabla w + 6\pi\rho U[\mathbf{Z}] \cdot w) - \int_{\mathbb{R}^3} (\nabla u[\rho,j] : \nabla w + 6\pi\rho u[\rho,j] \cdot w), \\ &= \int_{\mathbb{R}^3} (\nabla U[\mathbf{Z}] : \nabla w + 6\pi\rho U[\mathbf{Z}] \cdot w) - 6\pi \int_{\mathbb{R}^3} j \cdot w. \end{aligned}$$

We now work on the gradient term involved in this error:

$$\int_{\mathbb{R}^3} \nabla U[\mathbf{Z}] : \nabla w,$$

in the spirit of [14]. Applying the construction in [14, Appendix B], we obtain a covering $(T_\kappa)_{\kappa \in \mathbb{Z}^3}$ of $\mathbb{R}^3$ with cubes of width $\lambda[\mathbf{Z}]$ such that, denoting

$$\mathcal{Z}_\delta := \left\{ i \in \{1, \ldots, N\} \text{ s.t. } \mathrm{dist}\left(X_i \, , \bigcup_{\kappa \in \mathbb{Z}^3} \partial T_\kappa \right) < \frac{\lambda[\mathbf{Z}]}{\delta} \right\},$$

there holds:

$$\frac{1}{N} \sum_{i \in \mathcal{Z}_\delta} (1 + |V_i|^2) \leq \frac{12}{\delta} \frac{1}{N} \sum_{i=1}^N (1 + |V_i|^2). \tag{4.5}$$

Moreover, keeping only the indices $\mathcal{K}$ such that T_κ intersects the $1/N$ neighborhood of Ω_0, we obtain a covering $(T_\kappa)_{\kappa \in \mathcal{K}}$ of the $1/N$-neighborhood of Ω_0. We do not make precise the set of indices $\mathcal{K}$. The only relevant property to our computations is that

$$\#\mathcal{K} \lesssim \frac{|\Omega_1|}{|\lambda|^3}. \tag{4.6}$$

Associated with this covering, we introduce the following notations. For arbitrary $\kappa \in \mathcal{K}$, we set

$$\mathcal{I}_\kappa := \{ i \in \{1, \ldots, N\} \text{ s.t. } X_i \in T_\kappa \}, \quad M_\kappa[\mathbf{Z}] := \#\mathcal{I}_\kappa.$$

We note that, since T_κ has width $\lambda[\mathbf{Z}]$, we have that $M_\kappa[\mathbf{Z}] \leq M[\mathbf{Z}]$ for all κ. Moreover, by construction of $\mathcal{K}$, all the X_i are included in one T_κ so that the $(\mathcal{I}_\kappa)_{\kappa\in\mathcal{K}}$ realizes a partition of $\{1,\dots,N\}$.

We construct then an approximate test-function w^s piecewisely on the covering of Ω_0. Given $\kappa \in \mathcal{K}$, we set:

$$w^s_\kappa(x) = \sum_{i\in\mathcal{I}_\kappa\setminus\mathcal{Z}_\delta} G^N[w(X_i)](x-X_i)\,, \quad \forall\, x\in\mathbb{R}^3\,, \tag{4.7}$$

where $G^N[v]$ is the unique weak solution to the Stokes problem outside the unit ball with vanishing condition at infinity and constant boundary condition equal to $v\in\mathbb{R}^3$ on the unit ball. Explicit formulas are available in textbooks and are recalled in Appendix B. We set:

$$w^s = \sum_{\kappa\in\mathcal{K}} w^s_\kappa \mathbf{1}_{T_\kappa}.$$

We note that $w^s \notin H^1_0(\mathbb{R}^3)$ because of jumps at interfaces ∂T_κ. It will be sufficient for our purpose that $w^s \in H^1(\mathring{T}_\kappa)$ for arbitrary $\kappa\in\mathcal{K}$. Setting:

$$E_0[w] := \int_{\mathbb{R}^3} \nabla U[\mathbf{Z}] : \nabla w - \sum_{\kappa\in\mathcal{K}} \int_{\mathbb{R}^3} \nabla U[\mathbf{Z}] : \nabla w^s_\kappa,$$

we have:

$$E[w] = E_0[w] + \sum_{\kappa\in\mathcal{K}} \int_{T_\kappa} \nabla U[\mathbf{Z}] : \nabla w^s_\kappa + 6\pi \int_{\mathbb{R}^3} \rho U[\mathbf{Z}]\cdot w - 6\pi \int_{\mathbb{R}^3} j\cdot w.$$

Now for arbitrary $\kappa\in\mathcal{K}$, we apply in **Section 4.3** the properties of G^N and integrate by parts the integral on T_κ. We obtain an integral ∂T_κ in which we approximate $U[\mathbf{Z}]$ by:

$$\bar{u}_\kappa := \frac{1}{|[T_\kappa]_{2\delta}|} \int_{[T_\kappa]_{2\delta}} U[\mathbf{Z}](x)\mathrm{d}x,$$

where $[T_\kappa]_{2\delta}$ is the $\lambda[\mathbf{Z}]/(2\delta)$-neighborhood of ∂T_κ inside $\mathring{T}_\kappa$. In this way we obtain that

$$\int_{T_\kappa} \nabla U[\mathbf{Z}] : \nabla w^s_\kappa = \frac{6\pi}{N} \sum_{i\in\mathcal{I}_\kappa\setminus\mathcal{Z}_\delta} w(X_i)\cdot(V_i - \bar{u}_\kappa) + Err_\kappa.$$

where it will arise that Err_κ is due to the approximation of $U[\mathbf{Z}]$ by $\bar{u}_\kappa$ on ∂T_κ only. So, we set:

$$E_1[w] = \sum_{\kappa\in\mathcal{K}} \left(\int_{T_\kappa} \nabla U[\mathbf{Z}] : \nabla w^s_\kappa - \frac{6\pi}{N} \sum_{i\in\mathcal{I}_\kappa\setminus\mathcal{Z}_\delta} w(X_i)\cdot(V_i-\bar{u}_\kappa) \right)$$

and we rewrite:

$$\begin{aligned} E[w] = E_0[w] + E_1[w] + \frac{6\pi}{N}\sum_{\kappa\in\mathcal{K}}\sum_{i\in\mathcal{I}_\kappa\setminus\mathcal{Z}_\delta} w(X_i)\cdot V_i - \frac{6\pi}{N}\sum_{\kappa\in\mathcal{K}}\sum_{i\in\mathcal{I}_\kappa\setminus\mathcal{Z}_\delta} w(X_i)\cdot\bar{u}_\kappa \\ + 6\pi\int_{\mathbb{R}^3} \rho U[\mathbf{Z}]\cdot w - 6\pi\int_{\mathbb{R}^3} j\cdot w. \end{aligned}$$

Eventually, we obtain:

$$E[w] = E_0[w] + E_1[w] - E_\rho[w] + E_j[w], \tag{4.8}$$

where we denote:

$$E_j[w] := \frac{6\pi}{N}\sum_{\kappa\in\mathcal{K}}\sum_{i\in\mathcal{I}_\kappa\setminus\mathcal{Z}_\delta} w(X_i)\cdot V_i - 6\pi\int_{\mathbb{R}^3} j\cdot w,$$

$$E_\rho[w] := \sum_{\kappa\in\mathcal{K}} \left[\frac{6\pi}{N} \sum_{i\in\mathcal{I}_\kappa\setminus\mathcal{Z}_\delta} w(X_i) \right]\cdot\bar{u}_\kappa - 6\pi\int_{\mathbb{R}^3} \rho U[\mathbf{Z}]\cdot w.$$

Applying successively Lemma 4.4 , Lemma 4.5, Lemma 4.6 and Lemma 4.7 below, and recalling (4.3) to replace $\lambda[\mathbf{Z}], d_{\min}[\mathbf{Z}]$ and $M[\mathbf{Z}]$, we obtain respectively:

$$|E_0[w]| \lesssim \frac{1}{\eta}\left(\frac{1}{\delta} + \frac{1}{N^{\frac{2}{5}(1-\alpha)}} + \frac{\delta}{N^{\frac{4}{5}\alpha-\frac{2}{15}}}\right)^{\frac{1}{2}}\left(1+\frac{1}{N}\sum_{i=1}^N |V_i|^2\right)\|w\|_{D^2(\mathbb{R}^2)},$$

$$|E_1[w]| \lesssim \frac{\delta^6}{\sqrt{\eta}}\frac{1}{N^{\frac{2+3\alpha}{15}}}\left(1+\frac{1}{N}\sum_{i=1}^N |V_i|^2\right)^{\frac{1}{2}}\|w\|_{D^2(\mathbb{R}^2)},$$

$$|E_j[w]| \lesssim \left(\|j[\mathbf{Z}]-j\|_{[C_b^{0,1/2}(\mathbb{R}^3)]^*} + \frac{1}{\delta}\left(1+\frac{1}{N}\sum_{i=1}^N |V_i|^2\right)\right)\|w\|_{D^2(\mathbb{R}^2)},$$

$$|E_\rho[w]| \lesssim \left[\frac{1}{\sqrt{\delta}\sqrt{\eta}} + \frac{\|\rho\|_{L^2(\Omega_0)}}{\delta} + \delta^{\frac{9}{2}}\left(\frac{1}{N^{\frac{2+3\alpha}{15}}} + \|\rho[\mathbf{Z}]-\rho\|_{[C_b^{0,1/2}(\mathbb{R}^3)]^*}\right)\right]\left(1+\frac{1}{N}\sum_{i=1}^N |V_i|^2\right)^{\frac{5}{4}}\|w\|_{D^2(\mathbb{R}^3)}.$$

Gathering the above estimates, recalling that $\eta \in (0,1)$, $\delta > 1/2$, and remarking that, since $2/3 \leqslant \alpha < 1$ there holds

$$\frac{1-\alpha}{5} < \frac{2}{5}\left(\alpha - \frac{1}{3}\right) < \frac{2+3\alpha}{15},$$

we finally obtain:

$$|E[w]| \lesssim \frac{1}{\eta}\Bigg[\left(\frac{(1+\|\rho\|_{L^2(\Omega_0)})}{\sqrt{\delta}} + \delta^6\left(\frac{1}{N^{\frac{1}{5}(1-\alpha)}} + \|\rho[\mathbf{Z}]-\rho\|_{[C_b^{0,1/2}(\mathbb{R}^3)]^*}\right)\right)\left(1+\frac{1}{N}\sum_{i=1}^N |V_i|^2\right)^{\frac{5}{4}}$$
$$+ \|j[\mathbf{Z}]-j\|_{[C_b^{0,1/2}(\mathbb{R}^3)]^*}\Bigg]\|w\|_{D^2(\mathbb{R}^3)},$$

which ends the proof of Theorem 4.3. □

We proceed now to estimate the different error terms $E_0[w]$, $E_1[w]$, $E_j[w]$ and $E_\rho[w]$ appearing in the proof of Theorem 4.3 above. This is done in Sections 4.2, 4.3, 4.4 and 4.5, respectively.

4.2. **Estimating** $E_0[w]$**.** We recall that, with the notations above, there holds:

$$E_0[w] = \sum_{\kappa\in\mathcal{K}}\left(\int_{T_\kappa}\nabla U[\mathbf{Z}] : \nabla w - \int_{T_\kappa}\nabla U[\mathbf{Z}] : \nabla w_\kappa^s\right),$$

We have the following result:

Lemma 4.4. *For $N \geq 1$, we have:*

$$(4.9)\quad |E_0[w]| \lesssim \left(1+\frac{1}{N}\sum_{i=1}^N |V_i|^2\right)\dots$$
$$\dots\left(\frac{1}{\delta}\left(1+\frac{M[\mathbf{Z}]^2}{|Nd_{\min}[\mathbf{Z}]|^2} + \frac{M[\mathbf{Z}]^{\frac{5}{3}}}{Nd_{\min}[\mathbf{Z}]} + \frac{M[\mathbf{Z}]^2}{|Nd_{\min}[\mathbf{Z}]|^4}\right) + \frac{M[\mathbf{Z}]}{Nd_{\min}[\mathbf{Z}]} + \delta\frac{M[\mathbf{Z}]^{\frac{5}{3}}}{N\lambda[\mathbf{Z}]}\right)^{\frac{1}{2}}\|w\|_{D^2(\mathbb{R}^3)}.$$

Proof. The proof is a simpler version of [14, Proposition 11] but keeping track of the dependencies on w of all constants. Below, we use symbol $\lesssim$ to denote inequalities with constants that do not depend on N and δ.

First, we construct an intermediate test-function similar to [14, pp. 25-26]. We recall here the ideas of the construction. For arbitrary $\kappa \in \mathcal{K}$, we consider the Stokes problem on $\mathring{T}_\kappa \setminus \bigcup_{i\in\mathcal{I}_\kappa\setminus\mathcal{Z}_\delta}\overline{B_i}$ with boundary conditions:

$$(4.10)\qquad \left\{\begin{array}{rcll} u(x) &=& w(x)\,, & \text{on } \partial B_i \text{ for } i\in\mathcal{I}_\kappa\setminus\mathcal{Z}_\delta\,,\\ u(x) &=& 0\,, & \text{on } \partial T_\kappa\,. \end{array}\right.$$

The analysis of this problem is done in Appendix B and yields a solution $\bar{w}_\kappa$. We keep the symbol $\bar{w}_\kappa$ to denote its extension to Ω (by w on the holes and by 0 outside $\mathring{T}_\kappa^N$). We obtain a divergence-free $\bar{w}_\kappa \in H^1(\mathbb{R}^3)$ having support in $\Omega_0 + B(0,1)$. We then add the $\bar{w}_\kappa$ into:

$$\bar{w} = \sum_{\kappa\in\mathcal{K}}\bar{w}_\kappa\,.$$

and correct the values of $\bar{w}$ on the B_i when $i \in \mathcal{Z}_\delta$ in order that it fits the same boundary conditions as w on the B_i, $i = 1, \ldots, N$. We introduce χ^N a truncation function such that $\chi^N = 1$ in $B(0, 1/N)$ and $\chi^N = 0$ outside $B(0, 2/N)$ and we denote:

$$\begin{aligned}\tilde{w} &= \sum_{i \in \mathcal{Z}_\delta} \left[\chi^N(\cdot - X_i) w - \mathfrak{B}_{X_i, \frac{1}{N}, \frac{2}{N}}[x \mapsto w(x) \cdot \nabla \chi^N(x - X_i)] \right] \\ &\quad + \prod_{i \in \mathcal{Z}_\delta} (1 - \chi^N(\cdot - X_i)) \bar{w} + \sum_{i \in \mathcal{Z}_\delta} \mathfrak{B}_{X_i, \frac{1}{N}, \frac{2}{N}}[x \mapsto \bar{w}(x) \cdot \nabla \chi^N(x - X_i)] \,.\end{aligned}$$

where $\mathfrak{B}_{X, r_1, r_2}$ is the Bogovskii operator that lifts the divergence in bracket with a vector-field in $H^1_0(B(X, r_2) \setminus \overline{B(X, r_1)})$. Consequently, $w - \tilde{w} \in H^1_0(\mathcal{F})$ is an available test-function in the weak-formulation of the Stokes problem satisfied by $U[\mathbf{Z}]$. This yields:

$$\int_{\mathbb{R}^3} \nabla U[\mathbf{Z}] : \nabla(w - \tilde{w}) = 0.$$

We rewrite this identity as follows:

$$E_0[w] = \epsilon_1 + \epsilon_2 \,, \tag{4.11}$$

with:

$$\epsilon_1 = \sum_{\kappa \in \mathcal{K}} \int_{T_\kappa} \nabla U[\mathbf{Z}] : \nabla(\bar{w}_\kappa - w^s_\kappa) \,, \qquad \epsilon_2 = \int_{\Omega_1} \nabla U[\mathbf{Z}] : \nabla(\tilde{w} - \bar{w}) \,.$$

We control now the error term ϵ_1. For arbitrary $\kappa \in \mathcal{K}$, we apply Proposition B.1 to $\bar{w}_\kappa$ and we obtain:

$$\|\nabla(w^s_\kappa - \bar{w}_\kappa)\|_{L^2(T_\kappa)} \lesssim \frac{M_\kappa[\mathbf{Z}]}{N} \left(\frac{1}{d_{\min}[\mathbf{Z}]} + \frac{\delta}{\lambda[\mathbf{Z}]} \right)^{1/2} \left(\|w\|_{C^{0,1/2}(T_\kappa)} + \|\nabla w\|_{L^6(T_\kappa)} \right) \,.$$

Introducing this bound in the computation of ϵ_1 and recalling the two properties of $M_\kappa[\mathbf{Z}]$:

$$\sum_{\kappa \in \mathcal{K}} M_\kappa[\mathbf{Z}] \leq N \,, \qquad \sup_{\kappa \in \mathcal{K}} M_\kappa \leq M[\mathbf{Z}] \,, \tag{4.12}$$

yield:

$$|\epsilon_1| \lesssim \left(\frac{M[\mathbf{Z}]}{N d_{\min}[\mathbf{Z}]} + \delta \frac{M[\mathbf{Z}]}{N \lambda[\mathbf{Z}]} \right)^{\frac{1}{2}} \|\nabla U[\mathbf{Z}]\|_{L^2(\mathbb{R}^3)} \|w\|_{D^2(\mathbb{R}^3)} \,. \tag{4.13}$$

We compute now a bound for ϵ_2. For this, we replace $\tilde{w}$ by its explicit construction. We recall that the supports of the $(\chi^N(\cdot - X_i))_{i \in \{1, \ldots, N\}}$ are disjoint so that:

$$1 - \prod_{i \in \mathcal{Z}_\delta} (1 - \chi^N(x - X_i)) = \sum_{i \in \mathcal{Z}_\delta} \chi^N(x - X_i) \,, \quad \forall\, x \in \mathbb{R}^3.$$

Consequently, we split:

$$\begin{aligned}\bar{w} - \tilde{w} &= \sum_{i \in \mathcal{Z}_\delta} \left[\chi^N(\cdot - X_i) \bar{w} - \mathfrak{B}_{X_i, \frac{1}{N}, \frac{2}{N}}[x \mapsto \bar{w}(x) \cdot \nabla \chi^N(x - X_i)] \right] \\ &\quad - \sum_{i \in \mathcal{Z}_\delta} \left[\chi^N(\cdot - X_i) w - \mathfrak{B}_{X_i, \frac{1}{N}, \frac{2}{N}}[x \mapsto w(x) \cdot \nabla \chi^N(x - X_i)] \right] \,.\end{aligned}$$

and $\nabla(\bar{w} - \tilde{w}) = \sum_{i \in \mathcal{Z}_\delta} \sum_{\ell=1}^3 \epsilon^{(\ell)}_{2,i}$ where, for $i \in \mathcal{Z}_\delta$, we denote:

$$\begin{aligned}\epsilon^{(1)}_{2,i} &= -\nabla \left[\chi^N(\cdot - X_i) w - \mathfrak{B}_{X_i, \frac{1}{N}, \frac{2}{N}}[x \mapsto w(x) \cdot \nabla \chi^N(x - X_i)] \right] \,, \\ \epsilon^{(2)}_{2,i} &= \nabla \chi^N(\cdot - X_i) \otimes \bar{w} - \nabla \mathfrak{B}_{X_i, \frac{1}{N}, \frac{2}{N}}[x \mapsto \bar{w}(x) \cdot \nabla \chi^N(x - X_i)], \\ \epsilon^{(3)}_{2,i} &= \chi^N(\cdot - X_i) \nabla \bar{w}.\end{aligned}$$

We remark here that $\epsilon^{(\ell)}_{2,i}$ has support in $B(X_i, 2/N)$ whatever the value of ℓ. As previously, a standard Cauchy-Schwarz argument yields:

$$|\epsilon_2| \lesssim \|\nabla U[\mathbf{Z}]\|_{L^2(\mathbb{R}^3)} \left(\sum_{\ell=1}^3 \sum_{i \in \mathcal{Z}_\delta} |\epsilon^{(\ell)}_{2,i}|^2 \right)^{\frac{1}{2}} \,. \tag{4.14}$$

To complete the proof, it remains to bound the last term in the right-hand side of the above inequality.

First, by applying standard homogeneity properties of the Bogovskii operator and explicit computations, we have, for $i \in \mathcal{Z}_\delta$:

$$\begin{aligned}\int_{B(X_i,2/N)} |\epsilon_{2,i}^{(1)}|^2 &\leq \frac{1}{N}\|w\|^2_{L^\infty(\Omega_1)} + \|\nabla w\|^2_{L^2(B(X_i,2/N))}\\ &\lesssim \frac{1}{N}\left(\|w\|^2_{L^\infty(\Omega_1)} + \|\nabla w\|^2_{L^6(B(X_i,2/N))}\right).\end{aligned}$$

But, by the choice of the covering (see (4.5)), we have:

$$\sharp \mathcal{Z}_\delta \lesssim \frac{N}{\delta}\left(1 + \frac{1}{N}\sum_{i=1}^N |V_i|^2\right), \tag{4.15}$$

so, we obtain finally:

$$\sum_{i\in\mathcal{Z}_\delta}\int_{B(X_i,2/N)} |\epsilon_{2,i}^{(1)}|^2 \leq \frac{1}{\delta}\left(1 + \frac{1}{N}\sum_{i=1}^N |V_i|^2\right)\|w\|^2_{D^2(\mathbb{R}^3)}. \tag{4.16}$$

Secondly, with similar arguments as for $\epsilon_{2,i}^{(1)}$, we obtain, for $i \in \mathcal{Z}_\delta$:

$$\int_{B(X_i,2/N)} |\epsilon_{2,i}^{(2)}|^2 \lesssim N^2 \|\bar{w}\|^2_{L^2(B(X_i,2/N))}$$

and

$$\begin{aligned}\sum_{i\in\mathcal{Z}_\delta}\int_{B(X_i,2/N)} |\epsilon_{2,i}^{(2)}|^2 &\lesssim N^2 \sum_{i\in\mathcal{Z}_\delta}\sum_{\kappa\in\mathcal{K}} \|\bar{w}\|^2_{L^2(B(X_i,\frac{2}{N})\cap T_\kappa)},\\ &\lesssim N^2 \sum_{\kappa\in\mathcal{K}}\sum_{i\in\mathcal{Z}_\delta} \|\bar{w}_\kappa - w^s_\kappa\|^2_{L^2(B(X_i,\frac{2}{N})\cap T_\kappa)} + N^2 \sum_{i\in\mathcal{Z}_\delta}\sum_{\kappa\in\mathcal{K}} \|w^s_\kappa\|^2_{L^2(B(X_i,\frac{2}{N})\cap T_\kappa)}.\end{aligned}$$

We compute the first term on the last right-hand side thanks to the expansion (B.4) of G^N and remarking that, since the diameter of $B(X_i, \frac{2}{N})$ is infinitely smaller than the one of T_κ for N sufficiently large, one $B(X_i, 2/N)$ intersects at most 8 distinct T_κ. Repeating (4.15), we conclude:

$$\begin{aligned}\sum_{i\in\mathcal{Z}_\delta}\sum_{\kappa\in\mathcal{K}} \|w^s_\kappa\|^2_{L^2(B(X_i,\frac{2}{N})\cap T_\kappa)} &\lesssim \sum_{i\in\mathcal{Z}_\delta} 8\sup_{\kappa\in\mathcal{K}} \|w^s_\kappa\|^2_{L^2(B(X_i,\frac{2}{N}))},\\ &\lesssim \frac{|M[\mathbf{Z}]|^2}{N^4 d^2_{\min}[\mathbf{Z}]}\frac{1+\frac{1}{N}\sum_{i=1}^N|V_i|^2}{\delta}\|w\|^2_{L^\infty(\Omega_1)}.\end{aligned}$$

As for the other term, we introduce, for $\kappa \in \mathcal{K}$, the set $\mathcal{Z}_{\delta,\kappa}$ of indices i such that $B(X_i, \frac{2}{N}) \cap T_\kappa \neq \emptyset$, and we obtain, by repeated use of Hölder's inequality, that:

$$\begin{aligned}\sum_{\kappa\in\mathcal{K}}\sum_{i\in\mathcal{Z}_\delta} \|\bar{w}_\kappa - w^s_\kappa\|^2_{L^2(B(X_i,\frac{2}{N})\cap T_\kappa)} &= \sum_{\kappa\in\mathcal{K}}\sum_{i\in\mathcal{Z}_{\delta,\kappa}} \|\bar{w}_\kappa - w^s_\kappa\|^2_{L^2(B(X_i,\frac{2}{N})\cap T_\kappa)}\\ &\lesssim \sum_{\kappa\in\mathcal{K}} \frac{|\sharp\mathcal{Z}_{\delta,\kappa}|^{\frac{2}{3}}}{N^2}\|(\bar{w}_\kappa - w^s_\kappa)\|^2_{L^6(T_\kappa)}\\ &\lesssim \frac{1}{N^2}\left[\sum_{\kappa\in\mathcal{K}}\sharp\mathcal{Z}_{\delta,\kappa}\right]^{\frac{2}{3}}\left(\sum_{\kappa\in\mathcal{K}}\|(\bar{w}_\kappa - w^s_\kappa)\|^6_{L^6(T_\kappa)}\right)^{\frac{1}{3}}.\end{aligned}$$

By comparing the size of T_κ and $B(X_i, 2/N)$, we obtain again that:

$$\left[\sum_{\kappa\in\mathcal{K}}\sharp\mathcal{Z}_{\delta,\kappa}\right]^{\frac{2}{3}} \lesssim |\sharp\mathcal{Z}_\delta|^{\frac{2}{3}} \lesssim \left[\frac{N}{\delta}\left(1+\frac{1}{N}\sum_{i=1}^N|V_i|^2\right)\right]^{\frac{2}{3}}.$$

which, combined with Proposition B.1 and (4.12), yields:

$$\sum_{\kappa\in\mathcal{K}}\sum_{i\in\mathcal{Z}_\delta}\|(\bar{w}_\kappa - w^s_\kappa)\|^2_{L^2(B_\infty(X_i,\frac{2}{N})\cap T_\kappa)} \lesssim \frac{(1+\frac{1}{N}\sum_{i=1}^N|V_i|^2)^{\frac{2}{3}}}{\delta^{\frac{2}{3}}N^2}\frac{|M[\mathbf{Z}]|^{5/3}}{N}\left(\frac{1}{d_{\min}[\mathbf{Z}]}+\frac{\delta}{\lambda[\mathbf{Z}]}\right)\|w\|^2_{D^2(\mathbb{R}^3)}.$$

Combining the above inequalities and recalling (4.4), we conclude that:

$$\sum_{i\in\mathcal{Z}_\delta}\int_{B(X_i,2/N)}|\epsilon^{(2)}_{2,i}|^2 \lesssim \left(1+\frac{1}{N}\sum_{i=1}^N|V_i|^2\right)\dots \tag{4.17}$$
$$\dots\left(\frac{1}{\delta}\frac{|M[\mathbf{Z}]|^2}{|Nd_{\min}[\mathbf{Z}]|^2}+\frac{1}{\delta^{2/3}}\frac{|M[\mathbf{Z}]|^{5/3}}{Nd_{\min}[\mathbf{Z}]}+\delta^{1/3}\frac{|M[\mathbf{Z}]|^{5/3}}{N\lambda[\mathbf{Z}]}\right)\|w\|^2_{D^2(\mathbb{R}^3)}.$$

Finally, we have similarly:

$$\begin{aligned}\sum_{i\in\mathcal{Z}_\delta}\int_{B(X_i,2/N)}|\epsilon^{(3)}_{2,i}|^2 &\lesssim \sum_{i\in\mathcal{Z}_\delta}\sum_{\kappa\in\mathcal{K}}\|\nabla\bar{w}\|^2_{L^2(B(X_i,2/N)\cap T_\kappa)}\\ &\lesssim \sum_{\kappa\in\mathcal{K}}\sum_{i\in\mathcal{Z}_\delta}\|\nabla\bar{w}_\kappa-\nabla w^s_\kappa\|^2_{L^2(B(X_i,\frac{2}{N})\cap T_\kappa)}+\sum_{i\in\mathcal{Z}_\delta}\sum_{\kappa\in\mathcal{K}}\|\nabla w^s_\kappa\|^2_{L^2(B(X_i,\frac{2}{N})\cap T_\kappa)}.\end{aligned}$$

and we can reproduce the previous arguments relying on Proposition B.1. This yields, on the one hand:

$$\sum_{i\in\mathcal{Z}_\delta}\sum_{\kappa\in\mathcal{K}}\|\nabla w^s_\kappa\|^2_{L^2(B(X_i,\frac{2}{N})\cap T_\kappa)}\lesssim\frac{M^2[\mathbf{Z}]}{\delta|Nd_{\min}[\mathbf{Z}]|^4}\left(1+\frac{1}{N}\sum_{i=1}^N|V_i|^2\right)\|w\|^2_{D^2(\mathbb{R}^3)},$$

and, on the other hand:

$$\begin{aligned}\sum_{\kappa\in\mathcal{K}}\sum_{i\in\mathcal{Z}_\delta}\|\nabla\bar{w}_\kappa-\nabla w^s_\kappa\|^2_{L^2(B(X_i,\frac{2}{N})\cap T_\kappa)} &\lesssim \sum_{\kappa\in\mathcal{K}}\|\nabla\bar{w}_\kappa-\nabla w^s_\kappa\|^2_{L^2(T_\kappa)}\\ &\lesssim \frac{M[\mathbf{Z}]}{N}\left(\frac{1}{d_{\min}[\mathbf{Z}]}+\frac{\delta}{\lambda[\mathbf{Z}]}\right)\|w\|^2_{D^2(\mathbb{R}^3)}.\end{aligned}$$

We obtain finally that:

$$\sum_{i\in\mathcal{Z}_\delta}\int_{B(X_i,2/N)}|\epsilon^{(3)}_{2,i}|^2\lesssim\left(1+\frac{1}{N}\sum_{i=1}^N|V_i|^2\right)\left(\frac{1}{\delta}\frac{M^2[\mathbf{Z}]}{|Nd_{\min}[\mathbf{Z}]|^4}+\frac{M[\mathbf{Z}]}{Nd_{\min}[\mathbf{Z}]}+\delta\frac{M[\mathbf{Z}]}{N\lambda[\mathbf{Z}]}\right)\|w\|^2_{D^2(\mathbb{R}^3)}. \tag{4.18}$$

Introducing (4.16), (4.17) and (4.18) into (4.14) yields:

$$|\epsilon_2|\lesssim\left(1+\frac{1}{N}\sum_{i=1}^N|V_i|^2\right)\dots \tag{4.19}$$
$$\dots\left(\frac{1}{\delta}\left(1+\frac{M[\mathbf{Z}]^2}{|Nd_{\min}[\mathbf{Z}]|^2}+\frac{M[\mathbf{Z}]^{\frac{5}{3}}}{Nd_{\min}[\mathbf{Z}]}+\frac{M[\mathbf{Z}]^2}{|Nd_{\min}[\mathbf{Z}]|^4}\right)+\frac{M[\mathbf{Z}]}{Nd_{\min}[\mathbf{Z}]}+\delta\frac{M[\mathbf{Z}]^{\frac{5}{3}}}{N\lambda[\mathbf{Z}]}\right)^{\frac{1}{2}}\|w\|_{D^2(\mathbb{R}^3)}.$$

We complete the proof by combining (4.13)-(4.19). □

4.3. **Estimating $E_1[w]$.** We proceed with the computation of $E_1[w]$ defined by:

$$E_1[w]=\sum_{\kappa\in\mathcal{K}}\left(\int_{T_\kappa}\nabla U[\mathbf{Z}]:\nabla w^s_\kappa-6\pi\sum_{i\in\mathcal{I}_\kappa\setminus\mathcal{Z}_\delta}w(X_i)\cdot(V_i-\bar{u}_\kappa)\right).$$

We control this error term with the following lemma:

Lemma 4.5. *Given $N\geq 1$, we have:*

$$|E_1[w]|\lesssim\delta^6\sqrt{\frac{M[\mathbf{Z}]}{N\lambda[\mathbf{Z}]}}\left(1+\frac{1}{N}\sum_{i=1}^N|V_i|^2\right)^{\frac{1}{2}}\|w\|_{D^2(\mathbb{R}^3)}.$$

Proof. For N sufficiently large and $\kappa\in\mathcal{K}$, let simplify at first:

$$\tilde{I}_\kappa:=\int_{T_\kappa}\nabla U[\mathbf{Z}]:\nabla w^s_\kappa\,.$$

By definition, we have that:

$$w^s_\kappa(x)=\sum_{i\in\mathcal{I}_\kappa\setminus\mathcal{Z}_\delta}G^N[w(X_i)](x-X_i)\,,\quad\forall\,x\in\mathbb{R}^3\,,$$

so that, introducing the associated pressures $x \mapsto P^N[w(X_i^N)](x - X_i^N)$, we obtain (after several integration by parts as depicted in [14, pp. 32-33]):

$$\tilde{I}_\kappa = \frac{6\pi}{N} \sum_{i \in \mathcal{I}_\kappa \setminus \mathcal{Z}_\delta} (w(X_i) \cdot V_i - w(X_i) \cdot \bar{u}_\kappa) + Err_\kappa \tag{4.20}$$

with:

$$Err_\kappa = \int_{\partial T_\kappa} \left\{ \sum_{i \in \mathcal{I}_\kappa \setminus \mathcal{Z}_\delta} \partial_n G^N[w(X_i)](\cdot - X_i) - P^N[w(X_i)](\cdot - X_i) n \right\} \cdot (U[\mathbf{Z}] - \bar{u}_\kappa) \mathrm{d}\sigma \, .$$

Summing over $\kappa \in \mathcal{K}$, we obtain that

$$\begin{aligned} \sum_{\kappa \in \mathcal{K}} \int_{T_\kappa} \nabla U[\mathbf{Z}] : \nabla w_\kappa^s &= \sum_{\kappa \in \mathcal{K}} \tilde{I}_\kappa \\ &= \sum_{\kappa \in \mathcal{K}} \frac{6\pi}{N} \sum_{i \in \mathcal{I}_\kappa \setminus \mathcal{Z}_\delta} (w(X_i) \cdot V_i - w(X_i) \cdot \bar{u}_\kappa) + \sum_{\kappa \in \mathcal{K}} Err_\kappa, \end{aligned}$$

and also:

$$E_1[w] = \sum_{\kappa \in \mathcal{K}} Err_\kappa .$$

For $\kappa \in \mathcal{K}$, we adapt (up to notations) the computations of [14, pp. 34-35]. The point here is to lift the boundary condition $U[\mathbf{Z}] - \bar{u}_\kappa$ via a standard truncation process in order to yield a divergence-free vector-field v which vanishes at a distance $\lambda[\mathbf{Z}]/(2\delta)$ of ∂T_κ. Applying that $(G[w(X_i)], P[w(X_i)])$ solves the Stokes equation on $[T_\kappa]_{2\delta}$ (since this subset contains no holes with index in $\mathcal{I}_\kappa \setminus \mathcal{Z}_\delta$) we obtain:

$$|Err_\kappa| \leq C_{\mathfrak{B}}[2\delta](1 + C_{PW}[2\delta]) \left\{ \sum_{i \in \mathcal{I}_\kappa \setminus \mathcal{Z}_\delta} \|\nabla G^N[w(X_i)](\cdot - X_i)\|_{L^2([T_\kappa]_{2\delta})} \right\} \|\nabla U[\mathbf{Z}]\|_{L^2([T_\kappa]_{2\delta})} \, , \tag{4.21}$$

where, denoting $A(0, 1 - 1/\delta, 1)$ the cubic annulus $]-1,1[^3 \setminus [-(1-1/\delta), 1/\delta]^3$, we used the symbols:

- $C_{\mathcal{B}}[\delta]$ for the norm of the Bogovskii operator $\mathfrak{B}_{0,1-1/\delta,1}$ seen as a continuous linear mapping $L^2_0(A(0, 1 - 1/\delta, 1)) \to H^1_0(A(0, 1 - 1/\delta, 1))$,
- $C_{PW}[\delta]$ for the constant of the Poincaré-Wirtinger inequality on $H^1(A(0, 1 - 1/\delta, 1))$.

The asymptotics of these constants when $\delta \to \infty$ are analyzed in Appendix C.

To bound the first term on the right-hand side of this inequality, we remark again that for any $i \in \mathcal{I}_\kappa \setminus \mathcal{Z}_\delta$ the minimum distance between X_i and $[T_\kappa]_{2\delta}$ is larger than $\lambda[\mathbf{Z}]/(2\delta)$. Hence, applying the explicit formula (B.4) of the Stokeslet $G^N[w(X_i)]$ we obtain that

$$\begin{aligned} \|\nabla G^N[w(X_i)](\cdot - X_i)\|_{L^2([T_\kappa]_{2\delta})} &\leq \left(\int_{\lambda[\mathbf{Z}]/(2\delta)}^\infty \frac{\mathrm{d}r}{N^2 r^2} \right)^{\frac{1}{2}} |w(X_i)| \\ &\leq \frac{\sqrt{2\delta}}{N\sqrt{\lambda[\mathbf{Z}]}} |w(X_i)| \, . \end{aligned}$$

Combining these computations for the (at most) M_κ indices $i \in \mathcal{I}_\kappa \setminus \mathcal{Z}_\delta$ entails that:

$$\sum_{i \in \mathcal{I}_\kappa \setminus \mathcal{Z}_\delta} \|\nabla G^N[w(X_i)](\cdot - X_i)\|_{L^2([T_\kappa]_{2\delta})} \lesssim \frac{M_\kappa}{N} \sqrt{\frac{2\delta}{\lambda[\mathbf{Z}]}} \|w\|_{L^\infty(\Omega_1)} \, . \tag{4.22}$$

Plugging (4.22) into (4.21) and recalling the fundamental properties (4.12) of M_κ we conclude that

$$|E_1[w]| \lesssim C_{\mathfrak{B}}[2\delta](1 + C_{PW}[2\delta]) \sqrt{\frac{2\delta M[\mathbf{Z}]}{N\lambda[\mathbf{Z}]}} \|\nabla U[\mathbf{Z}]\|_{L^2(\mathbb{R}^3)} \|w\|_{C^{0,1/2}(\overline{\Omega_1})} .$$

We conclude the proof of Lemma 4.5 by applying that $C_{\mathfrak{B}}[2\delta](1 + C_{PW}[2\delta]) \lesssim \delta^{11/2}$ (see Appendix C) and recalling (4.4). □

4.4. **Estimating $E_j[w]$.** We proceed with the error term

$$E_j[w] = \sum_{\kappa\in\mathcal{K}} \frac{6\pi}{N} \sum_{i\in\mathcal{I}_\kappa\setminus\mathcal{Z}_\delta} w(X_i)\cdot V_i - 6\pi\int_{\mathbb{R}^3} j\cdot w.$$

Lemma 4.6. *Given $N\geq 1$, there holds:*

$$|E_j[w]| \lesssim \left(\|j[\mathbf{Z}]-j\|_{[C_b^{0,1/2}(\mathbb{R}^3)]^*} + \frac{1}{\delta}\left(1+\frac{1}{N}\sum_{i=1}^N |V_i|^2\right)\right)\|w\|_{D^2(\mathbb{R}^3)}\,.$$

Proof. As $w\in C_c^\infty(\mathbb{R}^3)$ and $(T_\kappa)_{\kappa\in\mathcal{K}}$ is a covering of $\mathrm{Supp}(j[\mathbf{Z}])$ we have that:

$$\sum_{\kappa\in\mathcal{K}}\sum_{i\in\mathcal{I}_\kappa} w(X_i)\cdot V_i = \langle j[\mathbf{Z}], w\rangle.$$

Consequently, complementing the sum in E_j with the indices in $\mathcal{Z}_\delta$, we have:

$$E_j[w] = 6\pi\langle j[\mathbf{Z}]-j, w\rangle + \frac{6\pi}{N}\sum_{i\in\mathcal{Z}_\delta} w(X_i)\cdot V_i.$$

The first term on the right-hand side is estimated straightforwardly:

$$|\langle j[\mathbf{Z}]-j,w\rangle| \leq \|j[\mathbf{Z}]-j\|_{[C^{0,1/2}(\mathbb{R}^3)]^*}\|w\|_{C_b^{0,1/2}(\overline{\Omega_1})},$$

while repeating the proof of [14, Lemma 15], we obtain, for $N\geq 1$:

$$\left|\frac{6\pi}{N}\sum_{i\in\mathcal{Z}_\delta\cap\mathcal{I}_\delta} w(X_i^N)\cdot V_i^N\right| \lesssim \frac{1}{\delta}\left(1+\frac{1}{N}\sum_{i=1}^N|V_i|^2\right)\|w\|_{L^\infty(\Omega_1)},$$

which yields the expected result and completes the proof of Lemma 4.6. □

4.5. **Estimating $E_\rho[w]$.** We end up by estimating the remainder term

$$E_\rho[w] = \sum_{\kappa\in\mathcal{K}}\left[\frac{6\pi}{N}\sum_{i\in\mathcal{I}_\kappa\setminus\mathcal{Z}_\delta} w(X_i)\right]\cdot\bar{u}_\kappa - 6\pi\int_{\mathbb{R}^3}\rho U[\mathbf{Z}]\cdot w.$$

Lemma 4.7. *For N sufficiently large, there holds:*

$$\begin{aligned}|E_\rho[w]| \lesssim \Bigg(\frac{1}{\delta^{\frac52}}\left(\sqrt{\frac{M[\mathbf{Z}]}{N|\lambda[\mathbf{Z}]|^3}}+\|\rho\|_{L^2(\Omega_0)}\right)+\frac{1}{\sqrt{\delta}}\left(\frac{M[\mathbf{Z}]}{N|\lambda[\mathbf{Z}]|^3}\right)^{1/4}\\ +\delta^{9/2}\left(\lambda[\mathbf{Z}]+\|\rho[\mathbf{Z}]-\rho\|_{[C_b^{0,1/2}(\mathbb{R}^3)]^*}\right)\Bigg)\left(1+\frac{1}{N}\sum_{i=1}^N|V_i|^2\right)^{\frac54}\|w\|_{D^2(\mathbb{R}^3)}.\end{aligned}$$

Proof. The proof is adapted from [19, Proposition 3.7]. As previously, let first complete the sum by reintroducing the $\mathcal{Z}_\delta$ indices:

$$\frac{6\pi}{N}\sum_{\kappa\in\mathcal{K}}\sum_{i\in\mathcal{I}_\kappa\setminus\mathcal{Z}_\delta} w(X_i)\cdot\bar{u}_\kappa = \frac{6\pi}{N}\sum_{\kappa\in\mathcal{K}}\sum_{i\in\mathcal{I}_\kappa} w(X_i)\cdot\bar{u}_\kappa - \widetilde{E}rr \tag{4.23}$$

where:

$$\widetilde{E}rr = \frac{6\pi}{N}\sum_{\kappa\in\mathcal{K}}\sum_{i\in\mathcal{I}_\kappa\cap\mathcal{Z}_\delta} w(X_i)\cdot\bar{u}_\kappa.$$

We have then:

$$E_\rho[w] = \frac{6\pi}{N}\sum_{\kappa\in\mathcal{K}}\sum_{i\in\mathcal{I}_\kappa} w(X_i)\cdot\bar{u}_\kappa - 6\pi\int_{\Omega_1}\rho U[\mathbf{Z}]\cdot w - \widetilde{E}rr.$$

We remark that we may rewrite the first term on the right-hand side of this equality by introducing:

$$\sigma = \left(1-\left(1-\frac{1}{2\delta}\right)^3\right)^{-1}\frac{1}{N|\lambda[\mathbf{Z}]|^3}\sum_{\kappa\in\mathcal{K}}\left(\sum_{i\in\mathcal{I}_\kappa} w(X_i)\right)\mathbf{1}_{[T_\kappa]_{2\delta}}\,,$$

which yields

$$E_\rho[w] = 6\pi\int_{\Omega_1}[\sigma-\rho w]\cdot U[\mathbf{Z}] - \widetilde{E}rr.$$

Finally, we introduce $U_\delta[\mathbf{Z}] := U[\mathbf{Z}] * \zeta_{\delta^3}$ in this identity (in order to regularize $U[\mathbf{Z}]$ so that we may make the difference between $\rho[\mathbf{Z}]$ and ρ appear) where we recall that $(\zeta_n)_n$ is a sequence of mollifiers. We apply below that

$$\|U_\delta[\mathbf{Z}]\|_{C^{0,1}(\overline{\Omega_1})} \lesssim \delta^{\frac{9}{2}} \|\nabla U[\mathbf{Z}]\|_{L^2(\mathbb{R}^3)}, \qquad \|U[\mathbf{Z}]_\delta - U[\mathbf{Z}]\|_{L^2(\Omega_1)} \lesssim \frac{\|\nabla U[\mathbf{Z}]\|_{L^2(\mathbb{R}^3)}}{\delta^3}. \tag{4.24}$$

Indeed, by classical computations there holds

$$\|U_\delta[\mathbf{Z}]\|_{C^{0,1}(\overline{\Omega_1})} \lesssim \|U_\delta[\mathbf{Z}]\|_{L^\infty(\Omega_1)} + \|\nabla U_\delta[\mathbf{Z}]\|_{L^\infty(\Omega_1)} \lesssim \|\nabla U[\mathbf{Z}]\|_{L^2(\mathbb{R}^3)}(1 + \|\zeta_{\delta^3}\|_{L^2(\mathbb{R}^3)}),$$

which yields the first inequality, and moreover

$$\begin{aligned}
\|U_\delta[\mathbf{Z}] - U[\mathbf{Z}]\|^2_{L^2(\Omega_1)} &= \int \left| \int_{|z|\le 1} [U[\mathbf{Z}](x - \tfrac{z}{\delta^3}) - U[\mathbf{Z}](x)]\zeta(z)\mathrm{d}z \right|^2 \mathrm{d}x \\
&\lesssim \int \int_{|z|\le 1} |U[\mathbf{Z}](x - \tfrac{z}{\delta^3}) - U[\mathbf{Z}](x)|^2 \mathrm{d}z\mathrm{d}x \\
&\lesssim \frac{1}{\delta^6} \int \int_{|z|\le 1} |z|^2 \int_0^1 |\nabla U[\mathbf{Z}](x - t\tfrac{z}{\delta^3})|^2 \mathrm{d}t\mathrm{d}z\mathrm{d}x \\
&\lesssim \frac{1}{\delta^6} \|\nabla U[\mathbf{Z}]\|^2_{L^2(\mathbb{R}^3)},
\end{aligned}$$

which implies the second one.

This entails that:

$$E_\rho[w] = \check{E}rr + \widehat{E}rr - \widetilde{E}rr$$

where

$$\check{E}rr = 6\pi \int_{\Omega_1} (\sigma - \rho w) \cdot (U[\mathbf{Z}] - U_\delta[\mathbf{Z}]), \qquad \widehat{E}rr = 6\pi \int_{\Omega_1} (\sigma - \rho w) \cdot U_\delta[\mathbf{Z}].$$

We proceed by estimating these three error terms independently.

We first remark that the $(\mathcal{I}_\kappa)_{\kappa \in \mathcal{K}}$ form a partition of $\{1, \dots, N\}$. This entails that:

$$\|\sigma\|_{L^1(\Omega_1)} \le \sum_{\kappa \in \mathcal{K}} \frac{M_\kappa}{N} \|w\|_{L^\infty(\Omega_1)} \le \|w\|_{L^\infty(\Omega_1)}.$$

Straightforward computations imply also that:

$$\begin{aligned}
\|\sigma\|_{L^\infty(\Omega_1)} &\le \left(1 - \left(1 - \frac{1}{2\delta}\right)^3\right)^{-1} \frac{M[\mathbf{Z}]}{N|\lambda[\mathbf{Z}]|^3} \|w\|_{L^\infty(\Omega_1)} \\
&\lesssim \delta \frac{M[\mathbf{Z}]}{N|\lambda[\mathbf{Z}]|^3} \|w\|_{L^\infty(\Omega_1)}.
\end{aligned}$$

By interpolating the above inequalities to control the L^2-norm of σ and combining with (4.24), we deduce:

$$\begin{aligned}
|\check{E}rr| &\lesssim \left(\|\sigma\|_{L^2(\Omega_1)} + \|\rho\|_{L^2(\Omega_1)} \|w\|_{L^\infty(\Omega_1)}\right) \|U_\delta[\mathbf{Z}] - U[\mathbf{Z}]\|_{L^2(\Omega_1)} \\
&\lesssim \left(\sqrt{\delta \frac{M[\mathbf{Z}]}{N|\lambda[\mathbf{Z}]|^3}} + \|\rho\|_{L^2(\Omega_1)} \right) \frac{\|\nabla U[\mathbf{Z}]\|_{L^2(\mathbb{R}^3)}}{\delta^3} \|w\|_{L^\infty(\Omega_1)} \\
&\lesssim \frac{1}{\delta^{5/2}} \left(\sqrt{\frac{M[\mathbf{Z}]}{N|\lambda[\mathbf{Z}]|^3}} + \|\rho\|_{L^2(\Omega_1)} \right) \|\nabla U[\mathbf{Z}]\|_{L^2(\mathbb{R}^3)} \|w\|_{L^\infty(\Omega_1)}.
\end{aligned} \tag{4.25}$$

Then, we note that we may rewrite:

$$\widehat{E}rr = \frac{6\pi}{N} \sum_{\kappa \in \mathcal{K}} \sum_{i \in \mathcal{I}_\kappa} \int_{[T_\kappa]_{2\delta}} \frac{w(X_i) \cdot U_\delta[\mathbf{Z}](x)}{|[T_\kappa]_{2\delta}|} - 6\pi \int_{\Omega_1} \rho U_\delta[\mathbf{Z}] \cdot w$$

where we rewrite the first term:

$$\begin{aligned}
&\frac{6\pi}{N} \sum_{\kappa \in \mathcal{K}} \sum_{i \in \mathcal{I}_\kappa} \int_{[T_\kappa]_{2\delta}} \frac{w(X_i) \cdot U_\delta[\mathbf{Z}]}{|[T_\kappa]_{2\delta}|} \\
&\qquad = \frac{6\pi}{N} \sum_{i=1}^N w(X_i) \cdot U_\delta[\mathbf{Z}](X_i) + \frac{6\pi}{N} \sum_{\kappa \in \mathcal{K}} \sum_{i \in \mathcal{I}_\kappa} \int_{[T_\kappa]_{2\delta}} \frac{w(X_i) \cdot (U_\delta[\mathbf{Z}] - U_\delta[\mathbf{Z}](X_i))}{|[T_\kappa]_{2\delta}|}.
\end{aligned}$$

Because $U_\delta[\mathbf{Z}]$ is Lipschitz, and by the estimate (4.24) on its Lipschitz norm, we have:

$$\begin{aligned}\left|\frac{6\pi}{N}\sum_{\kappa\in\mathcal{K}}\sum_{i\in\mathcal{I}_\kappa}\int_{[T_\kappa]_{2\delta}}\frac{w(X_i)\cdot(U_\delta[\mathbf{Z}]-U_\delta[\mathbf{Z}](X_i))}{|[T_\kappa]_{2\delta}|}\right| &\lesssim \lambda[\mathbf{Z}]\|U_\delta[\mathbf{Z}]\|_{C^{0,1}(\overline{\Omega_1})}\|w\|_{L^\infty(\Omega_1)}\\ &\lesssim \delta^{9/2}\lambda[\mathbf{Z}]\|\nabla U[\mathbf{Z}]\|_{L^2(\mathbb{R}^3)}\|w\|_{L^\infty(\Omega_1)}.\end{aligned}$$

On the other hand, we have:

$$\frac{6\pi}{N}\sum_{i=1}^N w(X_i)\cdot U_\delta[\mathbf{Z}](X_i)-6\pi\int_{\Omega_1}\rho\, U_\delta[\mathbf{Z}]\cdot w = 6\pi\langle\rho[\mathbf{Z}]-\rho, w\cdot U_\delta[\mathbf{Z}]\rangle$$

so that, introducing again the control on the $C^{0,1}$-norm of $U_\delta[\mathbf{Z}]$, we derive:

$$\left|\frac{6\pi}{N}\sum_{i=1}^N w(X_i)\cdot U_\delta[\mathbf{Z}](X_i)-6\pi\int_{\Omega_1}\rho U_\delta[\mathbf{Z}]\cdot w\right|\lesssim \delta^{9/2}\|\nabla U[\mathbf{Z}]\|_{L^2(\mathbb{R}^3)}\|\rho[\mathbf{Z}]-\rho\|_{[C_b^{0,1/2}(\mathbb{R}^3)]^*}\|w\|_{C^{0,1/2}(\overline{\Omega_1})}.$$

We finally obtain

$$|\widehat{E}rr|\lesssim\delta^{9/2}\left(\lambda[\mathbf{Z}]+\|\rho[\mathbf{Z}]-\rho\|_{[C^{0,1/2}(\mathbb{R}^3)]^*}\right)\|\nabla U[\mathbf{Z}]\|_{L^2(\mathbb{R}^3)}\|w\|_{C^{0,1/2}(\overline{\Omega_1})},\tag{4.26}$$

which completes the proof for the term $\widehat{E}rr$.

For the remaining term, we introduce:

$$\tilde\sigma=\left(1-\left(1-\frac{1}{2\delta}\right)^3\right)^{-1}\frac{1}{N|\lambda[\mathbf{Z}]|^3}\sum_{\kappa\in\mathcal{K}}\left(\sum_{i\in\mathcal{I}_\kappa\cap\mathcal{Z}_\delta}|w(X_i)|\right)\mathbf{1}_{[T_\kappa]_{2\delta}}\,,$$

so that:

$$|\widetilde{E}rr|\le\int_{\Omega_1}\tilde\sigma(x)|U[\mathbf{Z}](x)|\mathrm{d}x\,.$$

With similar arguments as in the previous computations, we have, applying (4.5):

$$\|\tilde\sigma\|_{L^1(\Omega_1)}\le\frac{1}{N}\#\mathcal{Z}_\delta\|w\|_{L^\infty(\Omega_1)}\le\frac{1}{\delta}\|w\|_{L^\infty(\Omega_1)}\left(1+\frac{1}{N}\sum_{i=1}^N|V_i|^2\right).$$

Furthermore, we have:

$$\|\tilde\sigma\|_{L^\infty(\Omega_1)}\lesssim\delta\frac{M[\mathbf{Z}]}{N|\lambda[\mathbf{Z}]|^3}\|w\|_{L^\infty(\Omega_1)}.$$

Consequently, by interpolation, we obtain:

$$\|\tilde\sigma\|_{L^{\frac43}(\Omega_1)}\lesssim\frac{1}{\sqrt\delta}\left(\frac{M[\mathbf{Z}]}{N|\lambda[\mathbf{Z}]|^3}\right)^{1/4}\|w\|_{L^\infty(\Omega_1)}\left(1+\frac{1}{N}\sum_{i=1}^N|V_i|^2\right)^{\frac34}.$$

Applying Sobolev embedding $\dot H^1(\mathbb{R}^3)\subset L^4(\Omega_1)$ with (4.4) we conclude that:

$$|\widetilde{E}rr|\lesssim\frac{1}{\sqrt\delta}\left(1+\frac1N\sum_{i=1}^N|V_i|^2\right)^{\frac34}\left(\frac{M[\mathbf{Z}]}{N|\lambda[\mathbf{Z}]|^3}\right)^{1/4}\|w\|_{C^{0,1/2}(\overline{\Omega_1})}\|\nabla U[\mathbf{Z}]\|_{L^2(\mathbb{R}^3)}\,.\tag{4.27}$$

We conclude the estimate of $E_\rho[w]$ by adding up (4.25), (4.26), (4.27) and recalling (4.4). □

5. Proof of the main result

We are now able to prove our main result Theorem 1.1 as well as the Corollary 1.2.

We hence consider the framework of Theorem 1.1. The main idea is to split the expectation we want to estimate into two parts: one taking into account the non-concentrated configurations (which has been treated in Section 4), and the other taking into account the concentrated configurations (treated in Section 2).

Let us fix $\alpha\in(2/3,1)$, $\eta=\min(1/(2C_1e),1)$ (see Assumption A1 or Proposition 2.4 to remind the meaning of constant C_1) and $R>0$. Given $N\in\mathbb{N}^*$ we denote:

$$M_N=N^{\frac{3(1-\alpha)}{5}}\quad\text{and}\quad\lambda_N=\left(\frac{\eta M_N}{N}\right)^{1/3}.$$

We can then introduce the corresponding decomposition of configurations with N particles:

$$\mathcal{O}^N = \left(\mathcal{O}^N \setminus (\mathcal{O}^N_{\lambda_N,M_N} \cup \mathcal{O}^N_\alpha)\right) \cup \left(\mathcal{O}^N_{\lambda_N,M_N} \cup \mathcal{O}^N_\alpha\right).$$

We emphasize that, since $\eta < 1$, for any $\mathbf{Z}^N \in \mathcal{O}^N \setminus (\mathcal{O}^N_{\lambda_N,M_N} \cup \mathcal{O}^N_\alpha)$, the associated configuration satisfies (4.3).

5.1. **Proof of Theorem 1.1.** We want to compute the expectation of the distance with $u := u[\rho, j]$. We split the expectation into the non-concentrated configurations and the concentrated configurations as follows

$$\begin{aligned}\mathbb{E}\left[\|U_N[\mathbf{Z}^N] - u\|_{L^2(B(0,R))}\right] &= \mathbb{E}\left[\mathbf{1}_{\mathcal{O}^N\setminus(\mathcal{O}^N_{\lambda_N,M_N}\cup\mathcal{O}^N_\alpha)}(\mathbf{Z}^N)\,\|U_N[\mathbf{Z}^N] - u\|_{L^2(B(0,R))}\right] \\ &\quad + \mathbb{E}\left[\mathbf{1}_{\mathcal{O}^N_{\lambda_N,M_N}\cup\mathcal{O}^N_\alpha}(\mathbf{Z}^N)\,\|U_N[\mathbf{Z}^N] - u\|_{L^2(B(0,R))}\right] \\ &=: I_1 + I_2.\end{aligned}$$

Let us first estimate the term I_2. Since we have chosen η sufficiently small, Proposition 2.4 entails that:

$$\mathbb{P}(\mathbf{Z}^N \in \mathcal{O}^N_{\lambda_N,M_N} \cup \mathcal{O}^N_\alpha) \lesssim N^{-(3\alpha-2)} \to 0 \quad \text{when } N \to \infty.$$

Consequently, with Corollary 3.8 we obtain that:

$$\begin{aligned}\mathbb{E}\left[\mathbf{1}_{\mathbf{Z}^N\in\mathcal{O}^N_{\lambda_N,M_N}\cup\mathcal{O}^N_\alpha}\|\nabla U[\mathbf{Z}^N]\|_{L^2(\mathbb{R}^3)}\right] &\leq \frac{K}{N} + \mathbb{P}(\mathbf{Z}^N \in \mathcal{O}^N_{\lambda_N,M_N} \cup \mathcal{O}^N_\alpha)^{\frac12}\,\mathbb{E}\left[\frac1N\sum_{i=1}^N |V_i^N|^2\right]^{\frac12} \\ &\lesssim \frac{1}{N^{\frac{3\alpha-2}{2}}}.\end{aligned}$$

Finally we get

$$\begin{aligned} I_2 &\lesssim \mathbb{E}\left[\mathbf{1}_{\mathcal{O}^N_{\lambda_N,M_N}\cup\mathcal{O}^N_\alpha}(\mathbf{Z}^N)\,\|U_N[\mathbf{Z}^N]\|_{D(\mathbb{R}^3)}\right] + \mathbb{E}\left[\mathbf{1}_{\mathcal{O}^N_{\lambda_N,M_N}\cup\mathcal{O}^N_\alpha}(\mathbf{Z}^N)\,\|u\|_{L^2(B(0,R))}\right] \\ &\lesssim \frac{1}{N^{\frac{3\alpha-2}{2}}} + \mathbb{P}\left[\mathbf{Z}^N \in \mathcal{O}^N_{\lambda_N,M_N} \cup \mathcal{O}^N_\alpha\right] \lesssim \frac{1}{N^{\frac{3\alpha-2}{2}}},\end{aligned}$$

We now turn to the term I_1. For N sufficiently large, noting that $\|\rho^N[\mathbf{Z}^N] - \rho\|_{[C_b^{0,1/2}(\mathbb{R}^3)]^*} \leq 2$, we can apply Theorem 4.3 choosing

$$\delta = \left[\frac{1 + \|\rho\|_{L^2(\Omega_0)}}{\left(\frac{1}{N^{\frac{1-\alpha}{5}}} + \|\rho^N[\mathbf{Z}^N] - \rho\|_{[C_b^{0,1/2}(\mathbb{R}^3)]^*}\right)}\right]^{\frac{2}{13}}.$$

This yields that, for arbitrary $\mathbf{Z}^N \in \mathcal{O}^N \setminus (\mathcal{O}^N_{\lambda_N,M_N} \cup \mathcal{O}^N_\alpha)$, we have:

$$\begin{aligned}\|U_N[\mathbf{Z}^N] - u\|_{L^2(B(0,R))} \lesssim \left(1 + \frac1N\sum_{i=1}^N |V_i^N|^2\right)^{\frac54}\left(1 + \|\rho\|_{L^2(\Omega_0)}\right)^{\frac{12}{13}}\left(\frac{1}{N^{\frac{1-\alpha}{5}}} + \|\rho^N[\mathbf{Z}^N] - \rho\|_{[C_b^{0,1/2}(\mathbb{R}^3)]^*}\right)^{\frac{1}{13}} \\ + \|j^N[\mathbf{Z}^N] - j\|_{[C_b^{0,1/2}(\mathbb{R}^3)]^*}.\end{aligned}$$

Taking expectation and using the hypotheses of the theorem, this yields

$$\begin{aligned} I_1 &\lesssim \frac{(1 + \|\rho\|_{L^2(\Omega_0)})^{\frac{12}{13}}}{N^{\frac{1-\alpha}{65}}}\quad \mathbb{E}\left[\left(1 + \frac1N\sum_{i=1}^N |V_i^N|^2\right)^{\frac54}\right] \\ &\quad + \mathbb{E}\left[\left(1 + \frac1N\sum_{i=1}^N |V_i^N|^2\right)^{\frac54}\|\rho[\mathbf{Z}^N] - \rho\|^{\frac{1}{13}}_{[C_b^{0,1/2}(\mathbb{R}^3)]^*}\right] + \mathbb{E}\left[\|j[\mathbf{Z}] - j\|_{[C_b^{0,1/2}(\mathbb{R}^3)]^*}\right] \\ &\lesssim [M_5(F_1^N)]^{\frac12}\left(\frac{(1 + \|\rho\|_{L^2(\Omega_0)})^{\frac{12}{13}}}{N^{\frac{1-\alpha}{65}}} + \mathbb{E}\left[\|\rho^N[\mathbf{Z}^N] - \rho\|_{[C_b^{0,1/2}(\mathbb{R}^3)]^*}\right]^{\frac{1}{13}}\right) + \mathbb{E}\left[\|j^N[\mathbf{Z}^N] - j\|_{[C_b^{0,1/2}(\mathbb{R}^3)]^*}\right] \\ &\lesssim \mathbb{E}\left[\|\rho^N[\mathbf{Z}^N] - \rho\|_{[C_b^{0,1/2}(\mathbb{R}^3)]^*}\right]^{\frac{1}{13}} + \mathbb{E}\left[\|j^N[\mathbf{Z}^N] - j\|_{[C_b^{0,1/2}(\mathbb{R}^3)]^*}\right] + N^{-\frac{(1-\alpha)}{65}} \\ &\lesssim \mathbb{E}\left[W_1(\rho^N[\mathbf{Z}^N], \rho)\right]^{\frac{1}{39}} + \mathbb{E}\left[\|j^N[\mathbf{Z}^N] - j\|_{[C_b^{0,1}(\mathbb{R}^3)]^*}\right]^{\frac13} + N^{-\frac{(1-\alpha)}{65}},\end{aligned} \tag{5.1}$$

where we have used Lemma 2.2 in last line.

We complete the proof of (1.9) by gathering previous estimates, and the last part of the theorem immediately follows from it. □

5.2. **Proof of the Corollary 1.2.** Let f satisfy the hypotheses of Corollary 1.2. We shall construct here a sequence $(F^N)_{N\in\mathbb{N}^*}$ of symmetric probability measures on $\mathcal{O}^N$ that satisfy Assumption A1 and that is f-chaotic with quantitative estimates (in the sense of Definition 2.1), hence also satisfies Assumption A2 thanks to Lemma 2.3.

A classical way in statistical physics to construct chaotic probability measures in the phase space of a N-particle system is to take the N-tensor product of a probability measure on the phase space of one particle that we condition to the energy surface of the system. More precisely, given a probability measure f on $\Omega_0\times\mathbb{R}^3$ we define a probability measure $\Pi^N[f]$ on $\mathcal{O}^N$ by

$$\Pi^N[f](\mathrm{d}\mathbf{z}^N) := \mathcal{W}_N^{-1}(f)\,\mathbf{1}_{\mathbf{z}^N\in\mathcal{O}^N}\, f^{\otimes N}(\mathrm{d}\mathbf{z}^N), \tag{5.2}$$

where $\mathcal{W}_N(f)$ is the partition function

$$\mathcal{W}_N(f) := \int_{(\Omega_0\times\mathbb{R}^3)^N} \mathbf{1}_{\mathbf{z}^N\in\mathcal{O}^N}\, f^{\otimes N}(\mathrm{d}\mathbf{z}^N).$$

We now verify that the sequence $(\Pi^N[f])_{N\in\mathbb{N}^*}$ satisfies Assumption A1. We start with a technical remark:

Lemma 5.1. *For any $1\le m\le N$ and N large enough there holds*

$$1\le \mathcal{W}_N^{-1}(f)\mathcal{W}_{N-m}(f)\le (1-8c_0\,N^{-2}\|\rho\|_{L^\infty(\mathbb{R}^3)})^{-m}\le e^{16c_0mN^{-2}\|\rho\|_{L^\infty(\mathbb{R}^3)}},$$

where $c_0=|B_{\mathbb{R}^3}|$ is the volume of the unit ball in $\mathbb{R}^3$.

Proof. We have

$$\begin{aligned}
\mathcal{W}_{m+1}(f) &= \int_{(\mathbb{R}^3\times\mathbb{R}^3)^{m+1}} \mathbf{1}_{(z_1,\dots,z_m)\in\mathcal{O}^m}\left(\prod_{i=1}^m \mathbf{1}_{|x_i-x_{m+1}|>\frac{2}{N}}\right) f^{\otimes(m+1)}(z_1,\dots,z_m,z_{m+1})\,\mathrm{d}z_1\dots\mathrm{d}z_{m+1}\\
&= \int_{(\mathbb{R}^3\times\mathbb{R}^3)^m}\left\{\int_{\mathbb{R}^3\times\mathbb{R}^3}\prod_{i=1}^m\left(1-\mathbf{1}_{|x_i-x_{m+1}|\le\frac{2}{N}}\right) f(z_{m+1})\,\mathrm{d}z_{m+1}\right\}\mathbf{1}_{(z_1,\dots,z_m)\in\mathcal{O}^m}\, f^{\otimes m}(z_1,\dots,z_m)\,\mathrm{d}z_1\dots\mathrm{d}z_m\\
&\ge \int_{(\mathbb{R}^3\times\mathbb{R}^3)^m}(1-8mc_0N^{-3}\|\rho\|_{L^\infty(\mathbb{R}^3)})\mathbf{1}_{(z_1,\dots,z_m)\in\mathcal{O}^m}\, f^{\otimes m}(z_1,\dots,z_m)\,\mathrm{d}z_1\dots\mathrm{d}z_m,
\end{aligned}$$

We note here that, to pass from the second to the last line, we only remark that the indicator functions deletes at most m balls of radius $2/N$ in $\mathbb{R}^3$. From the last inequality, we deduce $\mathcal{W}_{m+1}(f)\ge\mathcal{W}_m(f)(1-8mc_0N^{-3}\|\rho\|_{L^\infty(\mathbb{R}^3)})$. We conclude the proof of the first claimed inequality by induction.

For the second inequality, observe that $x\mapsto 2x+\log(1-x)$ is nonnegative for $0\le x\le 1/2$, therefore for N large enough (so that $16c_0mN^{-2}\|\rho\|_{L^\infty(\mathbb{R}^3)}\le 1$) we get

$$(1-8c_0\,N^{-2}\|\rho\|_{L^\infty(\mathbb{R}^3)})^{-m}\le e^{16c_0mN^{-2}\|\rho\|_{L^\infty(\mathbb{R}^3)}}.$$

□

As a consequence we obtain the following bounds on $(\Pi^N[f])_{N\in\mathbb{N}^*}$:

Lemma 5.2. *Given N sufficiently large, for any $1\le m\le N$ there holds:*

$$\begin{aligned}
&\|\Pi^N_m[f]\|_{L^\infty_x L^1_v(\mathcal{O}^m)}\le e^{16c_0mN^{-2}\|\rho\|_{L^\infty(\mathbb{R}^3)}}\,\|\rho\|^m_{L^\infty(\mathbb{R}^3)},\\
&\||z_1|^{k_0}\Pi^N_1[f]\|_{L^1_xL^1_v(\mathbb{R}^3\times\mathbb{R}^3)}\le e^{16c_0N^{-2}\|\rho\|_{L^\infty(\mathbb{R}^3)}}\int_{\mathbb{R}^3\times\mathbb{R}^3}|z_1|^{k_0}f(z_1)\mathrm{d}z_1,\\
&\||v_1|^{k_0}\Pi^N_2[f]\|_{L^\infty_xL^1_v(\mathcal{O}^2)}\le e^{32c_0N^{-2}\|\rho\|_{L^\infty(\mathbb{R}^3)}}\|\rho\|_{L^\infty}\sup_{x_1\in\mathbb{R}^3}\int_{\mathbb{R}^3}|v_1|^{k_0}f(x_1,v_1)\mathrm{d}v_1,
\end{aligned}$$

where $\Pi^N_m[f]$ denotes the m-marginal of $\Pi^N[f]$.

Proof. We write

$$\begin{aligned}
f^N_m(z_1,\dots,z_m) &\le \mathcal{W}_N^{-1}(f)\mathbf{1}_{(z_1,\dots,z_m)\in\mathcal{O}^m}\, f^{\otimes m}(z_1,\dots,z_m)\int_{(\mathbb{R}^3\times\mathbb{R}^3)^{N-m}}\prod_{m+1\le i<j\le N}\mathbf{1}_{|x_i-x_j|>\frac{2}{N}}\prod_{j=m+1}^N f(z_j)\,\mathrm{d}z_j\\
&\le \mathcal{W}_N^{-1}(f)\mathcal{W}_{N-m}(f)\mathbf{1}_{(z_1,\dots,z_m)\in\mathcal{O}^m}\, f^{\otimes m}(z_1,\dots,z_m).
\end{aligned}$$

Each estimate then follows easily by using the bound of Lemma 5.1. □

This lemma shows that $(\Pi^N[f])_{N\in\mathbb{N}^*}$ satisfies Assumption A1. We shall prove now that $(\Pi^N[f])_{N\in\mathbb{N}^*}$ is f-chaotic with quantitative estimates, which hence implies that it satisfies Assumption A2. To this end, we recall that we denote $(\mathbf{Z}^N)_{N\in\mathbb{N}}$ a sequence of random variables on $\mathcal{O}^N$ with corresponding laws $(\Pi^N[f])_{N\in\mathbb{N}^*}$ and that proving that $(\Pi^N[f])_{N\in\mathbb{N}^*}$ is f-chaotic reduces to measuring the expectation of the Wasserstein W_1-distance between the empirical measure $\mu^N[\mathbf{Z}^N]$ and f. This is the content of the following lemma, from which Corollary 1.2 follows straightforwardly.

Lemma 5.3. *Consider the framework of Corollary 1.2. Let $(\mathbf{Z}^N)_{N\in\mathbb{N}^*}$ be a sequence of random variables on $\mathcal{O}^N$ with laws $(\Pi^N[f])_{N\in\mathbb{N}^*}$ defined by* (5.2)*. There holds*

$$\mathbb{E}[W_1(\rho^N[\mathbf{Z}^N],\rho)] \lesssim \frac{1}{N^{1/3}} \quad \text{and} \quad \mathbb{E}[W_1(\mu^N[\mathbf{Z}^N],f)] \lesssim \frac{1}{N^{1/6}}.$$

Proof. We shall only prove the second estimate, the first one being similar arguing with the random variable $\mathbf{X}^N$ on $\mathcal{O}_x^N$ (coming from $\mathbf{Z}^N=(\mathbf{X}^N,\mathbf{V}^N)$).

Let $(\mathbf{W}^N)_{N\in\mathbb{N}^*}$ be a i.i.d. sequence of random variables on $(\mathbb{R}^3\times\mathbb{R}^3)^N$ with common law f, and $\mu^N[\mathbf{W}^N]$ be the associated empirical measure. We split

$$W_1(\mu^N[\mathbf{Z}^N],f) \le W_1(\mu^N[\mathbf{W}^N],f) + \mathbf{1}_{\mathbf{W}^N\in\mathcal{O}^N} W_1(\mu^N[\mathbf{Z}^N],\mu^N[\mathbf{W}^N]) + \mathbf{1}_{\mathbf{W}^N\notin\mathcal{O}^N} W_1(\mu^N[\mathbf{Z}^N],\mu^N[\mathbf{W}^N]),$$

which implies

$$\begin{aligned}\mathbb{E}\left[W_1(\mu^N[\mathbf{Z}^N],f)\right] &\le \mathbb{E}\left[W_1(\mu^N[\mathbf{W}^N],f)\right] + \mathbb{E}\left[\mathbf{1}_{\mathbf{W}^N\in\mathcal{O}^N} W_1(\mu^N[\mathbf{Z}^N],\mu^N[\mathbf{W}^N])\right]\\ &\quad + \mathbb{P}\left[\mathbf{W}^N\notin\mathcal{O}^N\right]^{\frac12} \mathbb{E}\left[W_1(\mu^N[\mathbf{Z}^N],\mu^N[\mathbf{W}^N])\right]^{\frac12}.\end{aligned}$$

The first term on the right-hand side can be controlled by $N^{-1/6}$ thanks to [9, Theorem 1], since $\mathbf{W}^N$ is a i.i.d. sequence of common law f and using the fact that f has support included in $\Omega_0\times\mathbb{R}^3$ as well as a finite moment of order 5. The second term is bounded (up to a constant) by the first one, indeed

$$\begin{aligned}\mathbb{E}\left[\mathbf{1}_{\mathbf{W}^N\in\mathcal{O}^N} W_1(\mu^N[\mathbf{Z}^N],\mu^N[\mathbf{W}^N])\right] &= \int_{\mathcal{O}^N}\int_{\mathcal{O}^N} W_1(\mu^N[\mathbf{z}^N],\mu^N[\mathbf{w}^N])\,\frac{\mathbf{1}_{\mathbf{z}^N\in\mathcal{O}^N} f^{\otimes N}(\mathrm{d}\mathbf{z}^N)}{\mathcal{W}_N(f)}\,\mathbf{1}_{\mathbf{w}^N\in\mathcal{O}^N} f^{\otimes N}(\mathrm{d}\mathbf{w}^N)\\ &\le \mathcal{W}_N(f)^{-1}\,\mathbb{E}\left[W_1(\mu^N[\widetilde{\mathbf{W}}^N],\mu^N[\mathbf{W}^N])\right]\\ &\lesssim \mathbb{E}\left[W_1(\mu^N[\mathbf{W}^N],f)\right] + \mathbb{E}\left[W_1(\mu^N[\widetilde{\mathbf{W}}^N],f)\right],\end{aligned}$$

where $\widetilde{\mathbf{W}}^N$ is an independent copy of $\mathbf{W}^N$. Finally the third term is bounded by $N^{-1/2}$ since $\mathbb{P}\left[\mathbf{W}^N\notin\mathcal{O}^N\right]\lesssim N^{-1}$ (thanks to a similar argument as in Lemma 2.6) and

$$\mathbb{E}\left[W_1(\mu^N[\mathbf{Z}^N],\mu^N[\mathbf{W}^N])\right] \lesssim \mathbb{E}\left[M_2(\mu^N[\mathbf{Z}^N])\right] + \mathbb{E}\left[M_2(\mu^N[\mathbf{W}^N])\right] = M_2(\Pi_1^N[f]) + M_2(f),$$

which are uniformly bounded. □

Appendix A. Construction of w_i

This section is devoted to the proof of **Lemma 3.5** and **Lemma 3.6**. We recall first the frame of these results. We assume that $N\in\mathbb{N}$ is given and strictly positive in the whole section and we drop the parameter N in most of notations. We consider N balls B_i, $i=1,\dots,N$, of centers $(X_1,\dots,X_N)\in\mathbb{R}^{3N}$ and common radii $1/N$. We assume that $|X_i-X_j|>2/N$ for $j\neq i$ so that these balls are disjoint.

We begin with **Lemma 3.6** on the possible intersections of $(B(X_i,\frac{3}{2N}))_{i=1,\dots,N}$. We recall the statement of this lemma and give a proof:

Lemma A.1. *Let $i\in\{1,\dots,N\}$. Setting*

$$\mathcal{I}_i := \{j\in\{1,\dots,N\} \text{ s.t. } B(X_i,\tfrac{3}{2N})\cap B(X_j,\tfrac{3}{2N})\neq\emptyset\},$$

we have that $\mathcal{I}_i$ contains at most 16 *distinct indices.*

Proof. The idea of this proof is adapted from [17].

Let $i \in \{1,\dots,N\}$ be fixed. Without restriction we may assume that $i=1$ and $X_1=0$. For arbitrary $j \in \mathcal{I}_1$ we have that $B(X_j,\frac{3}{2N}) \cap B(0,\frac{3}{2N}) \neq \emptyset$. This entails that $|X_j| \leq 3/N$ and $B(X_j,\frac{1}{N}) \subset B(0,\frac{4}{N})$. As the $B(X_j,\frac{1}{N})$ are disjoint by assumption, we have then:

$$\frac{4\pi}{3N^3}|\mathcal{I}_1| = |\bigcup_{j\in\mathcal{I}_1} B(X_j,\tfrac{1}{N})| \leq |B(0,\tfrac{4}{N})| \leq \frac{4\pi}{3N^3}16.$$

This completes the proof. □

We proceed with **Lemma 3.5** that we recall with the notations of **Section 3**:

Lemma A.2. *Given $i \in \{1,\dots,N\}$, there exists $w_i \in D(\mathbb{R}^3)$ satisfying*

$$w_i = V_i \text{ on } B_i \text{ and } w_i = 0 \text{ on } B_j \text{ for } j \neq i\,, \tag{A.1}$$

$$\mathrm{Supp}(w_i) \subset B(X_i,\tfrac{3}{2N}), \tag{A.2}$$

$$\|\nabla w_i\|^2_{L^2(\mathbb{R}^3)} \leq C\frac{|V_i|^2}{N}\left(1+\sum_{j\neq i}\frac{\mathbf{1}_{|X_i-X_j|<\frac{5}{2N}}}{|X_i-X_j|-\dfrac{2}{N}}\right), \tag{A.3}$$

for a universal constant C.

The remainder of this section is devoted to the proof of this result. Without loss of generality, we assume that $i=1$ and $X_1=0$. We look for w_1 of the form:

$$w_1(x) = \tilde{w}_1(Nx), \quad \forall\, x \in \mathbb{R}^3. \tag{A.4}$$

To define the constraints to be satisfied by $\tilde{w}_1$, we introduce notations for the shape of the fluid domain after dilation. Namely, we denote:

$$\tilde{X}_i = NX_i, \qquad \tilde{B}_i = B(NX_i,1), \quad \forall\, i=1,\dots,N.$$

In particular, $\tilde{B}_1 = B(0,1)$. We want now to construct $\tilde{w}_1 \in D(\mathbb{R}^3)$ such that:

$$\tilde{w}_1 = V_1 \text{ on } \tilde{B}_1 \text{ and } \tilde{w}_1 = 0 \text{ on } \tilde{B}_j \text{ for } j>1\,, \tag{A.5}$$

$$\mathrm{Supp}(\tilde{w}_1) \subset B(0,\tfrac{3}{2}), \tag{A.6}$$

A natural candidate for $\tilde{w}_1$ is obtained by focusing on (A.6). Indeed, introducing a truncation function $\chi_0 \in C^\infty(\mathbb{R})$ which satisfies:

$$\chi_0(t) = \begin{cases} 1 & \text{if } t<1,\\ 0 & \text{if } t>1+h_0,\end{cases}$$

with $h_0 \in (0,1/2)$ to be fixed later on, we may set:

$$\tilde{w}_{1,0} = \nabla \times \left[\frac{V_1 \times x}{2}\chi_0(|x|)\right].$$

This candidate satisfies indeed $\tilde{w}_{1,0} \in \mathcal{D}(\mathbb{R}^3)$ with

$$\tilde{w}_{1,0} = V_1 \text{ on } \tilde{B}_1, \qquad \mathrm{Supp}(\tilde{w}_{1,0}) \subset B(0,1+h_0) \subset B(0,\tfrac{3}{2}),$$

However, it does not take into account the balls that are too close to $\tilde{B}_1$. To match the further condition on these balls, we modify our candidate.

For this, let fix $j \in \{1,\dots,N\}$. To describe the geometry between $\tilde{B}_1$ and $\tilde{B}_j$ we introduce a system of coordinates (x_1,x_2,x_3) such that x_3 corresponds to the coordinates directed along $e_3 = \tilde{X}_j/|\tilde{X}_j|$. The associated cylindrical coordinates read:

$$r = \sqrt{x_1^2+x_2^2}, \quad e_r = \frac{1}{\sqrt{x_1^2+x_2^2}}(x_1,x_2,0), \quad \forall\,(x_1,x_2,x_3) \in \mathbb{R}^3 \setminus \{x_3=0\}.$$

We remark that, in these coordinates, close to $(0,0,1)$ the boundary $\partial\tilde{B}_1$ is the graph of the function $(x_1,x_2) \mapsto \gamma_b(\sqrt{x_1^2+x_2^2})$ where:

$$\gamma_b(r) = \sqrt{1-r^2}\,, \quad \forall\, r \in (0,1).$$

Furthermore, denoting by $h_j = \mathrm{dist}(\tilde{B}_1,\tilde{B}_j)$, we have also that close to $(0,0,1+h_j)$, the boundary $\partial\tilde{B}_j$ is the graph of the function $(x_1,x_2) \mapsto \gamma_t(\sqrt{x_1^2+x_2^2})$ where:

$$\gamma_t(r) = 2+h_j-\sqrt{1-r^2}\,, \quad \forall\, r \in (0,1).$$

Given $\delta > 0$ we set, in these cylindrical coordinate:

$$\mathfrak{C}_j[\delta] := \{(x_1,x_2,x_3) \in \mathbb{R}^3 \text{ s.t. } r \in (0,\delta) \text{ and } x_3 \in (\gamma_b(r),\gamma_t(r))\},$$
$$\mathfrak{A}_j[\delta] := \{(x_1,x_2,x_3) \in \mathbb{R}^3 \text{ s.t. } r \in (\delta/2,\delta) \text{ and } x_3 \in (\gamma_b(r),\gamma_b(\delta/2))\}.$$

These notations are illustrated by Figure 1.

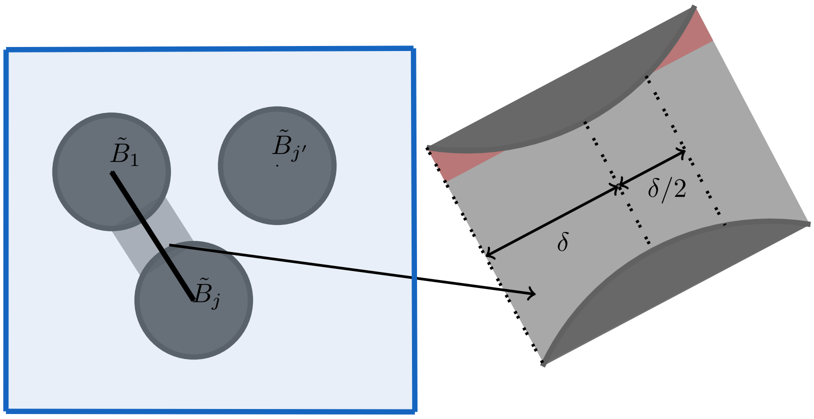

FIGURE 1. Notations $\mathfrak{A}_j[\delta]$ and $\mathfrak{C}_j[\delta]$
On the left a typical configuration is presented (in 2D). The gray zone corresponds to the set $\mathfrak{C}_j[\delta]$. On the right is a zoom on $\mathfrak{C}_j[\delta]$ where the subset $\mathfrak{A}_j[\delta]$ appears in the red color. We emphasize that the 3D geometry is obtained by revolution around the axis of the figure so that $\mathfrak{A}_j[\delta]$ is indeed connected.

We note that, whatever the value of $\delta \in (0,1)$ we have that $\mathfrak{C}_j[\delta]$ and $\mathfrak{A}_j[\delta]$ are Lipschitz, and that $\mathfrak{A}_j[\delta] \subset \mathfrak{C}_j[\delta]$. We have also the following technical property:

Proposition A.3. *There exists $h_{max} \in (0,1/2)$ and $\delta_0 \in (0,1/2)$ such that, if $h_j < h_{max}$ the following holds true:*

i) $\mathfrak{C}_j[\delta_0] \subset B(0,\frac{3}{2})$,

ii) $\overline{\mathfrak{C}_j[\delta_0]} \subset \mathbb{R}^3 \setminus \overline{\bigcup_{i\neq 1,j}^N \tilde{B}_i}$,

iii) for arbitrary $j' \neq j$ such that $h_{j'} < h_{max}$, there holds $\mathfrak{C}_j[\delta_0] \cap \mathfrak{C}_{j'}[\delta_0]$.

Proof. We compute restrictions on the values for δ_0 and h_{max} in order to fulfill the three conditions *i*), *ii*) and *iii*). This will yield an open set of admissible values for δ_0 and h_{max}.

For the proof, we only give two draws which explain where the restrictions come from. Let $j \in \{1,\dots,N\}$ such that $\mathrm{dist}(\tilde{B}_1,\tilde{B}_j) =: h_j < h_{max}$. In Figure 2, we illustrate that there exists a ball $\mathcal{V}_j$ centered in X_{j1} (the unique point in the closure of $\tilde{B}_j$ realizing the distance between $\tilde{B}_1$ and $\tilde{B}_j$) such that $\mathfrak{C}_j[\delta]$ (in blue on the figure) is contained in $\mathcal{V}_j$ (empty circle on the figure). The radius r_0 of this neighborhood is controlled by h_{max} and δ. In particular, for h_{max} and δ_0 sufficiently small we have $B(X_{j1},r_0) \subset B(0,1+h_{max}+r_0) \subset B(0,3/2)$ and *i*) is realized.

Second, we illustrate with Figure 3, that given another particle $\tilde{B}_{j'}$, the distance between $\tilde{B}_{j'}$ and the segment $[\tilde{X}_1,\tilde{X}_j]$ joining the centers of $\tilde{B}_1$ and $\tilde{B}_j$ is minimal when $\tilde{B}_{j'}$ is in simultaneous contact with $\tilde{B}_1$ and $\tilde{B}_j$ (several configurations are provided in red, the optimal one is the most opaque one). The minimal distance $r^{(j)}_{min}$ between $\tilde{B}_{j'}$ and $[\tilde{X}_1,\tilde{X}_j]$ is then a decreasing function of h_j vanishing when $h_j = 2(\sqrt{3}-1)$. The minimal distance $r^{(j,j')}$ between the point $X_{j'1}$ (the point in the closure of $\tilde{B}_{j'}$ realizing the distance with $\tilde{B}_1$) and X_{j1} is also realized with this configuration. It is then a continuous function of h_j which converges to 1 when $h_j \to 0$. So, with the

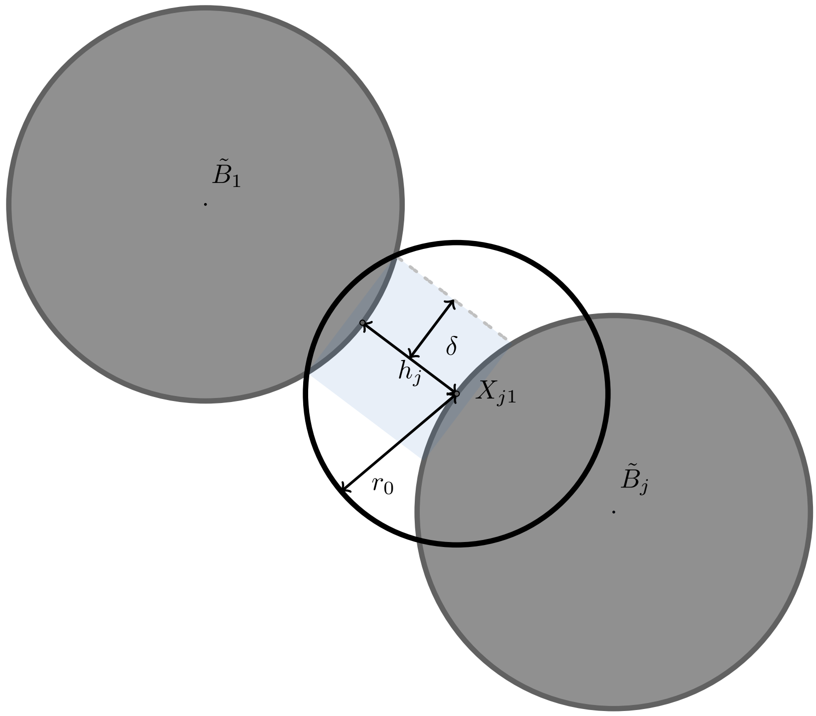

FIGURE 2. Construction of a neighborhood of X_{j1} containing $\mathfrak{C}_j$.

notations of the proof, for h_{max} and δ_0 small we have that $r_0 < r^{(j)}_{min}$ and $2r_0 < r^{(j,j')}$ so that $ii)$ and $iii)$ hold true.

□

With the proposition above, we can now fix h_{max}, δ_0 sufficiently small so that the conclusion of the proposition above holds true. Associated with δ_0 we set:

$$h_0 = \sqrt{\frac{\delta_0^2}{4} + \left(2 - \sqrt{1 - \left|\frac{\delta_0}{2}\right|^2}\right)^2} - 1.$$

If necessary, we restrict the size of δ_0 so that $h_0 < \min(1/2, h_{max})$. Associated with h_{max} we introduce:

$$\mathcal{J} := \left\{ j \in \{2, \dots, N\} \text{ s.t. } \mathrm{dist}(\tilde{B}_1, \tilde{B}_j) < h_{max} \right\} ,$$

We note that, by construction, we do have $h_0 > 0$ and that:

- since $h_0 < h_{max}$, $\tilde{w}_{1,0}$ vanishes on $\tilde{B}_i$ for $i \notin \mathcal{J}$.
- for $j \in \mathcal{J}$, χ_0 vanishes on $\partial\mathfrak{C}_j \cap \tilde{B}_j$ at a distance larger than $\delta_0/2$ from the axis $\mathbb{R}e_3$.

Furthermore, the $(\mathfrak{C}_j)_{j\in\mathcal{J}}$ are disjoint and do not intersect the $(\tilde{B}_i)_{i=1,\dots,N}$. So, in what follows, we construct $\tilde{w}_1$ on the $(\mathfrak{C}_j)_{j\in\mathcal{J}}$. We shall then extend $\tilde{w}_1$ by $\tilde{w}_{1,0}$ on the remaining fluid domain and by the expected values on the $(\tilde{B}_i)_{i=1,\dots,N}$.

Let $j \in \mathcal{J}$ and make precise $w_j = (\tilde{w}_1)_{|\mathfrak{C}_j}$. We decompose $w_j = w_j^{(1)} - w_j^{(2)}$. For $w_j^{(1)}$, we set:

$$w_j^{(1)}(x) = \nabla \times \left[\frac{V_1}{2} \times (x - e_3)\, \zeta_0(r) P\left(\frac{\gamma_t(r) - z}{\gamma_t(r) - \gamma_b(r)} \right) + (1 - \zeta_0(r))\chi_0(|x|) \frac{V_1}{2} \times x \right]$$

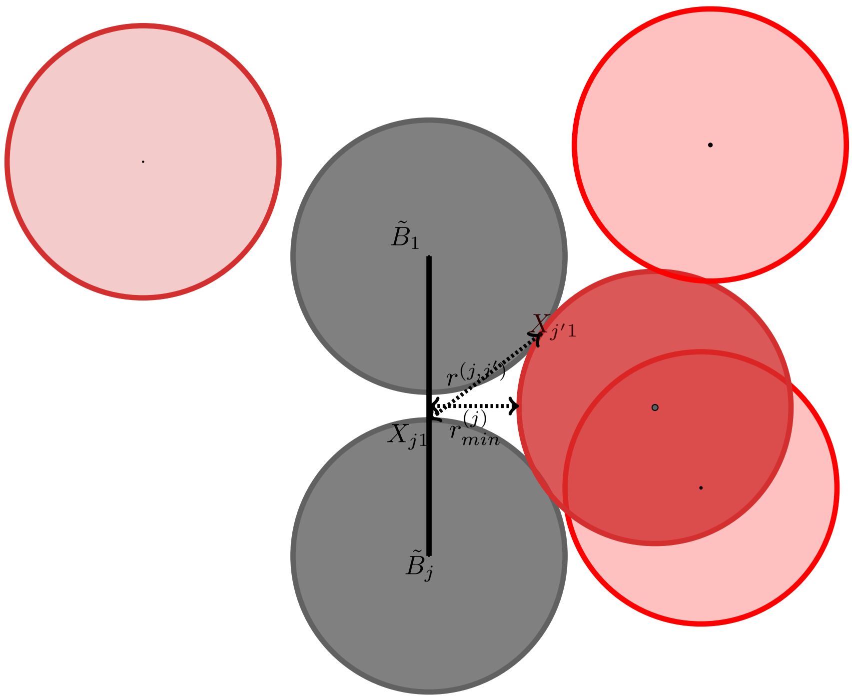

FIGURE 3. Minimizing configuration

where $P(t) = (3t^2 - 2t^3)$ for $t \in \mathbb{R}$ and $\zeta_0 \in C^\infty(\mathbb{R})$ is a truncation function such that:

$$\zeta_0(t) = \begin{cases} 1 & \text{if } t < \delta_0/2, \\ 0 & \text{if } t > 3\delta_0/4. \end{cases}$$

Clearly, we have that $w_j^{(1)} \in C^\infty(\overline{\mathfrak{C}_j})$ is divergence-free. Expanding the curl operator, we obtain:

$$w_j^{(1)}(x) = \begin{cases} 0 & \text{if } x \in \partial\mathfrak{C}_j \cap \partial\tilde{B}_j \ (i.e. z = \gamma_t(r)), \\ V_1 - \dfrac{\zeta_0'(r)}{2}(V_1 \times e_3) \times e_r & \text{if } x \in \partial\mathfrak{C}_j \cap \partial\tilde{B}_1 \ (i.e.\ z = \gamma_b(r)), \\ w_{1,0}(x) & \text{if } x \in \partial\mathfrak{C}_j \setminus \left(\partial\tilde{B}_1 \cup \partial\tilde{B}_j\right) \ (i.e.\ r = \delta). \end{cases} \tag{A.7}$$

All these identities derive from the choices for χ_0, ζ_0 and P. To obtain the first of these identities, it is worth noting that, with our choice for h_0, δ_0 the function $x \mapsto (1 - \zeta_0(r))\chi_0(r)$ vanishes on $\partial\tilde{B}_j \cap \partial\mathfrak{C}_j$.

Finally, we obtain that there exists a constant C_{max} depending only on (h_{max}, δ_0) such that:

$$\|\nabla w_j^{(1)}\|^2_{L^2(\mathfrak{C}_j)} \leq \frac{C_{max}|V_1|^2}{h_j}. \tag{A.8}$$

Indeed, away from the axis ($i.e.$ on $\mathfrak{C}_j \cap \{r > \delta_0/2\}$), $w_j^{(1)}$ depends smoothly on the parameter h_j. Hence, the contribution to $\|\nabla w_j^{(1)}\|_{L^2}$ is bounded by $C|V_1|^2$ where C is independent of h_j and depends only on δ_0, h_{max}. When $r < \delta_0/2$, we have:

$$w_j^{(1)}(x) = \nabla \times \left[\frac{V_1}{2} \times (x - e_3)\, P\left(\frac{z - \gamma_b(r)}{\gamma_t(r) - \gamma_b(r)}\right)\right]$$

Explicit computations show that, the worst term in $|\nabla w_j^{(1)}|$ corresponds to two differentiations of the P-term w.r.t. z, which we may bound by

$$|\partial_z w_j^{(1)}| \leq \frac{|V_1|}{2}\,|x-e_3|\left|\partial_{zz}P\left(\frac{z-\gamma_b(r)}{\gamma_t(r)-\gamma_b(r)}\right)\right| \leq C|V_1|(r+|z-\gamma_b(0)|)\frac{1}{(\gamma_t(r)-\gamma_b(r))^2}.$$

Remarking that $|z-\gamma_b(0)| \leq C|h_j+r|$ on $\mathfrak{C}_j$, we derive

$$\int_{\mathfrak{C}_j\cap\{r<\delta_0/2\}} |\nabla w_j^{(1)}(x)|^2 \mathrm{d}x \leq C|V_1|^2 \int_0^{\delta_0} \frac{|h_j+r|^2 r\mathrm{d}r}{(\gamma_t(r)-\gamma_b(r))^3}.$$

Combining then that $\gamma_t(r)-\gamma_b(r) \geq h_j + cr^2$ on $(0,\delta_0)$ for some $c>0$ (since $\delta_0<1/2$) and a change of variable $r=\sqrt{h_j}s$ in the integral, we obtain (A.8). More details on these computations can be found in [15].

In order that w_j fits the right boundary condition on $\partial\tilde{B}_1$, we add a corrector $w_j^{(2)}$ that compensate the error term that appears on the second line of (A.7), namely:

$$w_j^*(x) = \frac{\zeta_0'(r)}{2}\,[V_1\times e_3]\times e_r = \frac{\zeta_0'(r)}{2}(V_1\cdot e_r)e_3,$$

To construct $w_j^{(2)}$, we note that w_j^* is smooth and has compact support in $\partial\mathfrak{A}_j\cap\partial\tilde{B}_1$. Hence, we may extend w_j^* by 0 on $\partial\mathfrak{A}_j\setminus\partial\tilde{B}_1$. We obtain then a vector field $w_j^*\in C^\infty(\partial\mathfrak{A}_j)$ such that, by symmetry:

$$\int_{\partial\mathfrak{A}_j} w_j^*\cdot n\mathrm{d}\sigma = \int_{\partial\mathfrak{A}_j\cap\partial\tilde{B}_1} w_j^*(x)\cdot n\mathrm{d}\sigma = 0$$

Since, there exists a Bogovskii operator on the Lipschitz domain $\mathfrak{A}_j$, we construct $w_j^{(2)}\in H^1(\mathfrak{A}_j)$ such that:

(A.9) $$\operatorname{div} w_j^{(2)} = 0 \text{ in } \mathfrak{A}_j \quad w_j^{(2)} = w_j^* \text{ on } \partial\mathfrak{A}_j.$$

and such that:

$$\|w_j^{(2)}\|_{H^1(\mathfrak{A}_j)} \leq C\|w_j^*\|_{H^{1/2}(\partial\mathfrak{A}_j)}.$$

We note here that all the $\mathfrak{A}_j$ are isometric so that this last constant C is fixed by the values of δ_0 only and does not depend on j. Hence, there exists C_{max} depending only on δ_0 for which:

(A.10) $$\|w_j^{(2)}\|_{H^1(\mathfrak{A}_j)} \leq C_{max}|V_1|.$$

We note also that, on $\partial\mathfrak{A}_j$, w_j^* vanishes outside $\partial\mathfrak{A}_j\cap\partial\tilde{B}_1$ so that we may extend it by 0 on $\mathfrak{C}_j\setminus\mathfrak{A}_j$. We keep the same notations for simplicity. This yields a divergence-free vector-field $w_j^{(2)}\in H^1(\mathfrak{C}_j)$ defined on $\mathfrak{C}_j$.

By combination, it is then straightforward that $w_j = w_j^{(1)} - w_j^{(2)} \in H^1(\mathfrak{C}_j)$ satisfies:

i) $\operatorname{div} w_j = 0$ on $\mathfrak{C}_j$

ii) the following boundary conditions on $\partial\mathfrak{C}_j$:

$$w_j(x) = \begin{cases} 0 & \text{if } x\in\partial\mathfrak{C}_j\cap\partial\tilde{B}_j \\ V_1 & \text{if } x\in\partial\mathfrak{C}_j\cap\partial\tilde{B}_1 \\ w_{1,0}(x) & \text{if } x\in\partial\mathfrak{C}_j\setminus(\partial\tilde{B}_1\cup\partial\tilde{B}_j) \end{cases}$$

iii) the bounds (with a constant C_{max} depending only on δ_0, h_{max}):

$$\|\nabla w_j^{(1)}\|^2_{L^2(\mathfrak{C}_j)} \leq C_{max}|V_1|^2\left[1+\frac{1}{h_j}\right].$$

In particular, the above construction of $\tilde{w}_1$ on $\mathfrak{C}_j$ for fixed $j\in\mathcal{J}$, satisfies the right boundary conditions in order to extend it by $\tilde{w}_{1,0}$ on the remaining fluid domain. So, we set:

(A.11) $$\tilde{w}_1(x) = \begin{cases} V_1 & \text{if } x\in\tilde{B}_1 \\ w_j(x) & \text{if } x\in\mathfrak{C}_j,\ j\in\mathcal{J} \\ 0 & \text{if } x\in\tilde{B}_j,\ j\neq 1 \\ w_{1,0}(x) & \text{else.} \end{cases}$$

Combining (A.7)-(A.9) we obtain that $\tilde{w}_1\in H^1(\mathbb{R}^3)$ is divergence-free and satisfies the required conditions on the obstacles $(\tilde{B}_i)_{i=1,\dots,N}$. Furthermore, combining (A.7)-(A.10), we obtain a constant C_{max} depending only on

δ_0, h_{max} such that:

$$\|\nabla \tilde{w}_1\|^2_{L^2(\mathbb{R}^3)} \leq C_{max}|V_1|^2 \left(1 + \sum_{j\in\mathcal{J}} \frac{1}{h_j}\right) \leq C_{max}|V_1|^2 \left(1 + \sum_{j=2}^{N} \frac{\mathbf{1}_{|\tilde{X}_j|<5/2}}{|\tilde{X}_j| - 2}\right)$$

The associated vector-field w_1 (*via* the scaling (A.4)) satisfies then all the requirements of Lemma A.2.

Appendix B. Analysis of the cell problem

In this appendix, we fix $(N, M, \lambda) \in (\mathbb{N} \setminus \{0\})^2 \times (0, \infty)$, and a divergence-free $w \in C^\infty_c(\mathbb{R}^3)$. We denote T an open cube of width λ and $B_i = B(X_i, \frac{1}{N}) \subset T$ for $i = 1, \dots, M$. We assume further that there exists d_m satisfying

$$\min_{i=1,\dots,M} \left\{ \mathrm{dist}(X_i, \partial T), \min_{j\neq i} (|X_i - X_j|) \right\} \geq d_m > \frac{4}{N}\,. \tag{B.1}$$

We consider the Stokes problem:

$$\left\{ \begin{array}{rcl} -\Delta u + \nabla p &=& 0\,, \\ \mathrm{div}\, u &=& 0\,, \end{array} \right. \qquad \text{in } \mathcal{F} = T \setminus \bigcup_{i=1}^{M} \overline{B_i}\,, \tag{B.2}$$

completed with boundary conditions

$$\left\{ \begin{array}{rcll} u(x) &=& w(x)\,, & \text{in } B_i\,,\ \forall\, i = 1, \dots, M\,, \\ u(x) &=& 0\,, & \text{on } \partial T\,. \end{array} \right. \tag{B.3}$$

Assumption (B.1) entails that the B_i do not intersect and do not meet the boundary ∂T. So, the set $T \setminus \bigcup_{i=1}^{M} \overline{B_i}$ has a Lipschitz boundary that one can decompose in $M + 1$ connected components corresponding to ∂T and ∂B_i for $i = 1, \dots, M$.

For any $i = 1, \dots, M$, direct computations show that:

$$\int_{\partial B_i} w \cdot n \mathrm{d}\sigma = \int_{B_i} \mathrm{div}\, w = 0.$$

Hence, the problem (B.2)-(B.3) is solved by applying [14, Theorem 3] and it admits a unique generalized solution $u \in H^1(\mathcal{F})$. We want to compare this solution with:

$$u_s(x) = \sum_{i=1}^{M} G^N[w(X_i)](x - X_i),$$

where, for arbitrary $v \in \mathbb{R}^3$, $G^N[v]$ is the unique vector-field that can be associated to a pressure $P^N[v]$ in order to form a pair solution to the Stokes problem outside the unit ball. Explicit formulas for these solutions can be found in standard textbooks:

$$G^N[v](x) := \frac{1}{4N}\left(\frac{3}{|x|} + \frac{1}{N^2|x|^3}\right) v + \frac{3}{4N}\left(\frac{1}{|x|} - \frac{1}{N^2|x|^3}\right)\frac{v\cdot x}{|x|^2} x\,, \tag{B.4}$$

$$P^N[v](x) := \frac{3}{2N}\frac{v \cdot x}{|x|^3}\,. \tag{B.5}$$

The main result of this appendix section reads:

Proposition B.1. *There exists a constant K independent of (N, M, d_m, w, λ) for which:*

$$\|(u - u_s)\|_{L^6(\mathcal{F})} + \|\nabla(u - u_s)\|_{L^2(\mathcal{F})} \leq K\Big[\|w\|_{C^{0,1/2}(\overline{T})} + \|\nabla w\|_{L^6(T)}\Big]\sqrt{\frac{M}{N}}\left(\frac{1}{\sqrt{N}} + \sqrt{\frac{M}{d_m}}\right).$$

Proof. We split the error term into two pieces. First, we reduce the boundary conditions of the Stokes problem (B.2)-(B.3) to constant boundary conditions. Then, we compare the solution to the Stokes problem with constant boundary conditions to the combination of Stokeslets u_s. In the whole proof, the symbol $\lesssim$ is used when the implicit constant in the written inequality does not depend on N, M, d_m, w and λ.

So, we introduce u_c the unique generalized solution to the Stokes problem on $\mathcal{F}$ with boundary conditions:

$$\left\{ \begin{array}{rcll} u_c &=& w(X_i)\,, & \text{in } B_i\,,\ \forall\, i = 1, \dots, M\,, \\ u_c &=& 0\,, & \text{on } \partial T\,. \end{array} \right. \tag{B.6}$$

Again, existence and uniqueness of this velocity-field holds by applying [14, Theorem 3]. We split then:

$$\begin{aligned}
\|(u-u_s)\|_{L^6(\mathcal{F})} &\leq \|(u-u_c)\|_{L^6(\mathcal{F})} + \|(u_c-u_s)\|_{L^6(\mathcal{F})}\,,\\
\|\nabla(u-u_s)\|_{L^2(\mathcal{F})} &\leq \|\nabla(u-u_c)\|_{L^2(\mathcal{F})} + \|\nabla(u_c-u_s)\|_{L^2(\mathcal{F})}.
\end{aligned}$$

To control the first term on the right-hand sides, we note that $(u-u_c)$ is the unique generalized solution to the Stokes problem on $\mathcal{F}$ with boundary conditions:

$$\left\{\begin{array}{rcll}
(u-u_c)(x) &=& w(x)-w(X_i)\,, & \text{in } B_i\,,\ \forall\, i=1,\ldots,M\,,\\
(u-u_c)(x) &=& 0\,, & \text{on } \partial T\,.
\end{array}\right.$$

Hence, by the variational characterization of Theorem 3.2, $\|\nabla(u-u_c)\|_{L^2(\mathcal{F})}$ realizes the minimum of $\|\nabla\tilde{w}\|_{L^2(\mathcal{F})}$ amongst

$$\left\{\tilde{w}\in H^1(\mathcal{F}) \text{ s.t. } \operatorname{div}\tilde{w}=0\,,\ \tilde{w}_{|\partial T}=0\,,\ \tilde{w}_{|\partial B_i}=w(\cdot)-w(X_i)\,,\ \forall\, i=1,\ldots,M\right\}.$$

We construct thus a suitable $\tilde{w}$ in this space. We set:

$$\tilde{w}=\sum_{i=1}^{M}\tilde{w}_i$$

with, for $i=1,\ldots,M$:

$$\tilde{w}_i=\left(\chi^N(\cdot-X_i)(w(\cdot)-w(X_i))-\mathfrak{B}_{X_i,\frac{1}{N},\frac{2}{N}}\left[x\mapsto(w(x)-w(X_i))\cdot\nabla\chi^N(x-X_i)\right]\right).$$

In this definition χ^N is again chosen truncation function that between $B(0,\frac{1}{N})$ and $B(0,\frac{2}{N})$. We assume further that χ^N is obtained from χ^1 by dilation. The operator $\mathfrak{B}_{X_i,\frac{1}{N},\frac{2}{N}}$ denotes the Bogovskii operator on the annulus

$$A(X_i,\frac{1}{N},\frac{2}{N})=B(0,\tfrac{2}{N})\setminus\overline{B(0,\tfrac{1}{N})}.$$

The properties of this operator are analyzed in [14, Appendix A] (though these results are nowadays classical and can also be found in [1] for instance). It is straightforward to verify that the mean of $x\mapsto(w(x)-w(X_i))\cdot\nabla\chi^N(x-X_i)$ vanishes so that the above vector-field $\tilde{w}_i$ is well-defined. We note that $\tilde{w}_i$ has support in $B(X_i,\frac{2}{N})$ so that, as $d_m>4/N$, the $\tilde{w}_i$ have disjoint supports inside T. This yields that $\tilde{w}$ is indeed divergence-free and fits the required boundary conditions. Furthermore, there holds:

$$\|\nabla\tilde{w}\|_{L^2(\mathcal{F})}\leq\left[\sum_{i=1}^{M}\|\nabla\tilde{w}_i\|^2_{L^2(B(X_i,\frac{2}{N}))}\right]^{\frac{1}{2}}.$$

For $i\in\{1,\ldots,M\}$ we have by direct computations:

$$\begin{aligned}
\|\nabla\chi^N(\cdot-X_i)(w(\cdot)-w(X_i))\|^2_{L^2(B_\infty(X_i^N,\frac{2}{N}))} &\lesssim \frac{\|w\|^2_{C^{0,1/2}}}{N^2}\,,\\
\|\chi^N(\cdot-X_i)\nabla(w(\cdot)-w(X_i))\|^2_{L^2(B_\infty(X_i,\frac{2}{N}))} &\lesssim \frac{\|w\|^2_{W^{1,6}(T)}}{N^2}\,,
\end{aligned}$$

and, by applying [14, Lemma 16]:

$$\begin{aligned}
&\|\nabla\mathfrak{B}_{X_i^N,\frac{1}{N},\frac{2}{N}}\left[x\mapsto(w(x)-w(X_i))\cdot\nabla\chi^N(x-X_i)\right]\|^2_{L^2(B_\infty(X_i,\frac{2}{N}))}\\
&\qquad\lesssim \|x\mapsto(w(x)-w(X_i))\cdot\nabla\chi^N(x-X_i)\|^2_{L^2(B(X_i,\frac{2}{N}))}\\
&\qquad\lesssim \frac{\|w\|^2_{C^{0,1/2}(\overline{T})}}{N^2}.
\end{aligned}$$

Gathering all these inequalities in the computation of $\tilde{w}$ yields finally:

$$\|\nabla\tilde{w}\|_{L^2(\mathcal{F})}\lesssim\sqrt{M}\,\frac{\|w\|_{C^{0,1/2}(\overline{T})}+\|w\|_{W^{1,6}(T)}}{N}.$$

The variational characterization of generalized solutions to Stokes problems entails that we have the same bound for $(u-u_c)$. At this point, we argue that the straightforward extension of u and u_c (by w and $w(X_i)$ on the B_i

respectively) satisfy $(u-u_c) \in H^1_0(T) \subset L^6(T)$ so that

$$\begin{aligned}\|u-u_c\|_{L^6(\mathcal{F})} &\leq \|u-u_c\|_{L^6(T)} \lesssim \|\nabla(u-u_c)\|_{L^2(T)} \\ &\lesssim \left(\|\nabla(u-u_c)\|^2_{L^2(\mathcal{F})} + M\frac{\|w\|^2_{W^{1,6}(T)}}{N^2}\right)^{\frac12} \\ &\lesssim \sqrt{M}\frac{\|w\|_{C^{0,1/2}(\overline{T})} + \|w\|_{W^{1,6}(T)}}{N}.\end{aligned}$$

We emphasize that, by a scaling argument, the constant deriving from the embedding $H^1_0(T) \subset L^6(T)$ does not depend on λ so that it is not significant to our problem.

We turn to estimating $u_c - u_s$. Due to the linearity of the Stokes equations, we split

$$u_c = \sum_{i=1}^M u_{c,i},$$

where $u_{c,i}$ is the generalized solution to the Stokes problem on $\mathcal{F}$ with boundary conditions:

$$\begin{cases} u_{c,i} = w(X_i)\,, & \text{on } \partial B_i\,, \\ u_{c,i} = 0\,, & \text{on } \partial T \cup \bigcup_{j\neq i} \partial B_j\,. \end{cases}$$

We have then

$$\|\nabla(u_c - u_s)\|_{L^2(\mathcal{F})} \leq \sum_{i=1}^M \|\nabla(u_{c,i} - G^N[w(X_i)](\cdot - X_i))\|_{L^2(\mathcal{F})}. \tag{B.7}$$

Similarly, we expand :

$$u_s = \sum_{i=1}^M G_i\,, \text{ where } G_i(x) = G^N[w(X_i)](x-X_i)\,, \quad \forall\, x \in \mathbb{R}^3.$$

For $i \in \{1,\dots,M\}$ we extend $u_{c,i}$ by 0 on $\mathbb{R}^3 \setminus T$ and B_j for $j \neq i$. The extension we still denote by $u_{c,i}$ satisfies $u_{c,i} \in H^1(\mathbb{R}^3 \setminus \overline{B_i})$ and is divergence-free. In particular, we have $u_{c,i} \in D(\mathbb{R}^3 \setminus \overline{B_i})$. Consequently, $u_{c,i} - G_i \in D(\mathbb{R}^3 \setminus \overline{B_i})$ and:

$$\begin{aligned}\|\nabla(u_{c,i} - G_i)\|^2_{L^2(\mathcal{F})} &\leq \int_{\mathbb{R}^3\setminus\overline{B_i}} |\nabla u_{c,i} - \nabla G_i|^2 \\ &\leq \int_{\mathbb{R}^3\setminus\overline{B_i}} |\nabla u_{c,i}|^2 - 2\int_{\mathbb{R}^3\setminus\overline{B_i}} \nabla u_{c,i} : \nabla G_i + \int_{\mathbb{R}^3\setminus\overline{B_i}} |\nabla G_i|^2\,.\end{aligned}$$

To compute the product term, we apply that $u_{c,i}$ and $G_i = G^N[w(X_i)](\cdot - X_i)$ have the same trace on ∂B_i and that U_i is a generalized solution to the Stokes problem on $\mathbb{R}^3 \setminus \overline{B_i}$. So, integrals of the form $\int_{\mathbb{R}^3\setminus\overline{B_i}} \nabla G_i : \nabla w$ (for $w \in D(\mathbb{R}^3 \setminus \overline{B_i})$) depend only on the trace of w on ∂B_i. This entails that:

$$\int_{\mathbb{R}^3\setminus\overline{B_i}} \nabla u_{c,i} : \nabla G_i = \int_{\mathbb{R}^3\setminus\overline{B_i}} |\nabla G_i|^2\,,$$

and we have:

$$\|\nabla(u_{c,i} - G_i)\|^2_{L^2(\mathcal{F})} \leq \int_{\mathbb{R}^3\setminus\overline{B_i}} |\nabla u_{c,i}|^2 - \int_{\mathbb{R}^3\setminus\overline{B_i}} |\nabla G_i|^2\,. \tag{B.8}$$

To conclude, we find a bound from above for

$$\int_{\mathbb{R}^3\setminus\overline{B_i}} |\nabla u_{c,i}(x)|^2 \mathrm{d}x = \int_{\mathcal{F}} |\nabla u_{c,i}(x)|^2 \mathrm{d}x.$$

As $u_{c,i}$ is a generalized solution to a Stokes problem on $\mathcal{F}$, this can be done by constructing a divergence-free $\bar{w}_i$ satisfying the same boundary condition as $u_{c,i}$. We define:

$$\bar{w}_i = \chi_{d_m/4}(\cdot - X_i)G_i - \mathfrak{B}_{X_i,\frac{d_m}{4},\frac{d_m}{2}}\left[x \mapsto G_i(x)\cdot\nabla\chi_{d_m/4}(x-X_i)\right]$$

where $\chi_{d_m/4}$ truncates between $B(0,d_m/4)$ and $B(0,d_m/2)$. As previously, we have here a divergence-free function which satisfies the right boundary conditions because $\chi_{d_m/4}(\cdot - X_i) = 1$ on B_i (since $d_m/4 > 1/N$) and vanishes on all the other boundaries of $\partial\mathcal{F}$ (since the distance between one hole center and the other holes or ∂T

is larger than $d_m - 1/N > d_m/2$). Again, similarly as in the computation of $\tilde{w}_i$ we apply the properties of the Bogovskii operator $\mathfrak{B}_{X_i,\frac{d_m}{4},\frac{d_m}{2}}$ and there exists an absolute constant K for which:

$$\|\nabla \bar{w}_i\|^2_{L^2(\mathcal{F})} \leq \int_{\mathbb{R}^3\setminus\overline{B_i}} |\chi_{d_m/4}(\cdot - X_i)\nabla G_i|^2 + K\left(\int_{A(X_i,\frac{d_m}{4},\frac{d_m}{2})} |\nabla G_i(x)|^2 + |\nabla\chi_{d_m/4}(x-X_i)\otimes G_i(x)|^2 \mathrm{d}x\right).$$

As we have the same bound for $u_{c,i}$, we plug the right-hand side above into (B.8) and get:

$$\|\nabla(u_{c,i} - G_i)\|^2_{L^2(\mathcal{F})} \lesssim \int_{\mathbb{R}^3\setminus B(X_i,\frac{d_m}{4})} |\nabla G_i(x)|^2\mathrm{d}x + \int_{A(X_i,\frac{d_m}{4},\frac{d_m}{2})} |\nabla\chi_{d_m/4}(x-X_i)\otimes G_i(x)|^2\mathrm{d}x\,.$$

With the explicit decay properties for G_i (see (B.4)) and $\nabla\chi_{d_m/4}$ we derive:

$$\int_{\mathbb{R}^3\setminus B(X_i,\frac{d_m}{4})} |\nabla G_i(x)|^2\mathrm{d}x + \int_{A(X_i,\frac{d_m}{4},\frac{d_m}{2})} |\nabla\chi_{d_m/4}(x-X_i)\otimes G_i(x)|^2\mathrm{d}x \lesssim \frac{\|w\|^2_{L^\infty}}{N^2 d_m}\,.$$

Combining these bounds for $i = 1,\ldots,M$ in (B.7) we get:

$$\|\nabla(u_c - u_s)\|_{L^2(\mathcal{F})} \leq \frac{M\|w\|_{L^\infty(T)}}{N\sqrt{d_m}}.$$

By similar arguments, we also have:

$$\|u_c - u_s\|_{L^6(\mathcal{F})} = \|u_c - u_s\|_{L^6(T)} \leq \sum_{i=1}^{M} \|u_{c,i} - G_i\|_{L^6(\mathbb{R}^3\setminus\overline{B_i})}.$$

As $u_{c,i}, G_i \in D(\mathbb{R}^3 \setminus \overline{B_i})$ and $u_{c,i}, G_i$ share the same value on ∂B_i, there holds $u_{c,i} - G_i \in D_0(\mathbb{R}^3\setminus\overline{B_i})$ and we may use the classical inequality (see [10, (II.6.9)]):

$$\|u_{c,i} - G_i\|_{L^6(\mathbb{R}^3\setminus\overline{B_i})} \lesssim \|\nabla u_{c,i} - \nabla G_i\|_{L^2(\mathbb{R}^3\setminus\overline{B_i})}\,, \quad \forall\, i = 1,\ldots,M\,,$$

(again the constant arising from this embedding does not depend on N by a standard scaling argument). This yields again the bound:

$$\|(u_c - u_s)\|_{L^6(\mathcal{F})} \leq \frac{M\|w\|_{L^\infty(T)}}{N\sqrt{d_m}}.$$

Finally, combining the error terms between u_c and u_s and between u and u_c we obtain

$$\|(u-u_s)\|_{L^6(\mathcal{F})} + \|\nabla(u-u_s)\|_{L^2(\mathcal{F})} \leq K\sqrt{\frac{M}{N}}\left(\frac{1}{\sqrt{N}} + \sqrt{\frac{M}{Nd_m}}\right)\Big[\|w\|_{C^{0,1/2}(\overline{T})} + \|\nabla w\|_{L^6(T)}\Big].$$

This ends the proof. $\square$

We note that, when we apply Proposition B.1 in this article, we will choose $M \geq 1$ and d_m that has to be small. In that case we have that

$$\frac{1}{\sqrt{N}} \leq 2\sqrt{\frac{M}{Nd_m}},$$

and the result of Proposition B.1 reads:

$$\|(u-u_s)\|_{L^6(\mathcal{F})} + \|\nabla(u-u_s)\|_{L^2(\mathcal{F})} \leq K\Big[\|w\|_{C^{0,1/2}(\overline{T})} + \|\nabla w\|_{L^6(T)}\Big]\frac{M}{N\sqrt{d_m}}.$$

Appendix C. Analysis of some constants

In this section, we consider the problem of finding constants for the Poincaré-Wirtinger inequality and the Bogovskii operator on a cubic annulus $A(0, 1-1/\delta, 1) :=]-1,1[^3\setminus[-(1-1/\delta), 1-1/\delta]^3$. In both proofs, we proceed by change of variables (since only the asymptotics of the constant when $\delta \to \infty$ is needed). For this, we fix $\delta > 2$. We introduce a odd strictly increasing application $\chi_\delta \in C^2([-1,1])$ such that

$$\chi_\delta([0,1/2]) = [0, 1-1/\delta], \quad \chi_\delta(1) = 1.$$

For this, we introduce an even $\zeta \in C^\infty(\mathbb{R})$ such that:

$$\mathbf{1}_{[-1/4,1/4]} \leq \zeta \leq \mathbf{1}_{[-1/2,1/2]}.$$

We fix a constant k to be chosen later on and we define χ'_δ as the interpolation between $2(1-1/\delta)$ on [0,1/2] and k on $[1/2+1/\delta,1]$ that we integrate from $t=0$. This means:

$$\chi_\delta(x) = \int_0^x 2(1-1/\delta)\zeta(\delta(s-1/2)_+) + k(1-\zeta(\delta(s-1/2)_+)\mathrm{d}s.$$

With this choice, we fix k so that $\chi_\delta(1)=1$ yielding:

$$k = \frac{1-2(1-1/\delta)\int_0^1 \zeta(\delta(s-1/2)_+)\mathrm{d}s}{\int_0^1(1-\zeta(\delta(s-1/2)_+)\mathrm{d}s} = \frac{1-2(1-1/\delta)(1/2+\int_0^{1/\delta}\zeta(\delta s)\mathrm{d}s)}{\int_0^{1/2}(1-\zeta(\delta s))\mathrm{d}s} = O\left(\frac{1}{\delta}\right).$$

We emphasize that, due to our choice for ζ, we have $\int_0^1 \zeta(s)\mathrm{d}s < 1/2$. This entails that we have also $k>0$ and χ_δ is indeed strictly increasing.

Consequently, we have that:

- χ_δ realizes a C^2-diffeomorphism from $[-1,1]$ to $[-1,1]$ such that $\chi_\delta([-1/2,1/2]) = [-(1-1/\delta),1-1/\delta]$,
- $1/\delta \lesssim \chi'_\delta(y) \leq 2$ and $\chi''_\delta(y) \lesssim \delta$ for any $y\in[-1,1]$.

We introduce σ_δ its converse mapping. It satisfies:

- $\sigma_\delta([-(1-1/\delta),1-1/\delta]) = [-1/2,1/2]$,
- $1/2 \leq \sigma'_\delta(x) \leq \delta$ and $\sigma''_\delta(x) \lesssim \delta^4$ for any $x\in[-1,1]$.

Finally, we denote X_δ and Y_δ the corresponding C^2-diffeomorphisms between $A(0,1/2,1)$ and $A(0,1-1/\delta,1)$:

$$\begin{array}{rccc} X_\delta: & A(0,1/2,1) & \longrightarrow & A(0,1-1/\delta,1) \\ & (y_1,y_2,y_3) & \longmapsto & (\chi_\delta(y_1),\chi_\delta(y_2),\chi_\delta(y_3))) \end{array} \qquad \begin{array}{rccc} Y_\delta: & A(0,1-1/\delta,1) & \longrightarrow & A(0,1/2,1) \\ & (x_1,x_2,x_3) & \longmapsto & (\sigma_\delta(x_1),\sigma_\delta(x_2),\sigma_\delta(x_3))) \end{array}$$

We start with the Poincaré-Wirtinger inequality. Our main result reads:

Proposition C.1. *There holds $C_{PW}[\delta] \lesssim \delta$. Namely, given $f\in L^2_0(A(0,1-1/\delta,1))\cap H^1(A(0,1-1/\delta,1))$, we have:*

$$\int_{A(0,1-1/\delta,1)} |f(x)|^2\mathrm{d}x \lesssim \delta^2 \int_{A(0,1-1/\delta,1)} |\nabla f(x)|^2\mathrm{d}x. \tag{C.1}$$

Proof. We fix $f\in L^2_0(A(0,1-1/\delta,1))\cap H^1(A(0,1-1/\delta,1))$ and, with the previous notations, let us consider:

$$\tilde{f}(y) = f(X_\delta(y)) - \oint \tilde{f}, \qquad \forall\, y\in A(0,1/2,1),$$

with

$$\oint \tilde{f} := \int_{A(0,1/2,1)} f(X_\delta(y))\mathrm{d}y.$$

Standard computations show that $\tilde{f}\in L^2_0(A(0,1/2,1))\cap H^1(A(0,1/2,1))$ so that, by the Poincaré-Wirtinger inequality we have:

$$\int_{A(0,1/2,1)} |\tilde{f}(y)|^2\mathrm{d}y \lesssim \int_{A(0,1/2,1)} |\nabla\tilde{f}(y)|\mathrm{d}y.$$

Conversely, there holds:

$$f(x) = \tilde{f}(Y_\delta(x)) + \oint \tilde{f}, \qquad \forall\, x\in A(0,1-1/\delta,1).$$

Hence, because f is mean-free on $A(0,1-1/\delta,1)$, there holds:

$$\begin{aligned}\int_{A(0,1-1/\delta,1)} |f(x)|^2\mathrm{d}x &\leq \int_{A(0,1-1/\delta,1)} |f(x)|^2 + |A(0,1/2,1)|\left[\oint\tilde{f}\right]^2 \\ &\leq \int_{A(0,1-1/\delta,1)} \left|f(x) - \oint\tilde{f}\right|^2\mathrm{d}x \\ &\leq \int_{A(0,1-1/\delta,1)} |\tilde{f}(Y_\delta(x))|^2\mathrm{d}x\end{aligned}$$

We can then transform the geometry to go back in the $A(0,1/2,1)$ and apply the previous inequalities on σ'_δ :

$$\begin{aligned}\int_{A(0,1-1/\delta,1)} |f(x)|^2 \mathrm{d}x &\leq \left(\prod_{i=1}^{3} \max_{x_i\in[0,1]} \frac{1}{\sigma'_\delta(x_i)}\right) \int_{A(0,1-1/\delta,1)} |\tilde{f}(Y_\delta(x))|^2 \prod_{i=1}^{3} \sigma'_\delta(x_i)\mathrm{d}x_i \\ &\lesssim \int_{A(0,1/2,1)} |\tilde{f}(y)|^2 \mathrm{d}y \\ &\lesssim \int_{A(0,1/2,1)} |\nabla \tilde{f}(y)|^2 \mathrm{d}y.\end{aligned}$$

At this point, we compute $\nabla \tilde{f}$ with respect to ∇f and apply the previous inequalities on χ'_δ:

$$\begin{aligned}\int_{A(0,1/2,1)} |\nabla \tilde{f}(y)|^2 \mathrm{d}y &\lesssim \int_{A(0,1/2,1)} \sum_{i=1}^{3} \chi'_\delta(y_i)^2 |\partial_i f(X_\delta(y))|^2 \mathrm{d}y \\ &\lesssim \int_{A(0,1/2,1)} \sum_{i=1}^{3} \frac{\chi'_\delta(y_i)}{\prod_{j\neq i}\chi'_\delta(y_j)} |\partial_i f(X_\delta(y))|^2 \prod_{j=1}^{3} \chi'_\delta(y_j)\mathrm{d}y_j \\ &\lesssim \delta^2 \int_{A(0,1-1/\delta,1)} |\nabla f(x)| \mathrm{d}x.\end{aligned}$$

This ends the proof. ☐

Finally, we consider the Bogovskii operator on the annulus:

Proposition C.2. *There holds $C_{\mathfrak{B}}[\delta] \lesssim \delta^{9/2}$. Namely, given $f \in L^2_0(A(0,1-1/\delta,1))$ there exists $u \in H^1_0(A(0,1-1/\delta,1))$ such that*

$$\begin{aligned}&\operatorname{div} u = f \ \text{ on } A(0,1-1/\delta,1) \\ &\|\nabla u\|_{L^2(A(0,1-1/\delta,1))} \lesssim \delta^{9/2} \|f\|_{L^2(A(0,1-1/\delta,1))}\end{aligned}$$

Proof. We provide a proof by change of variable as for the previous proposition. Given $f \in L^2_0(A(0,1-1/\delta,1))$ we define

$$\hat{f}(y) = \prod_{i=1}^{3} \chi'_\delta(y_i) f(X_\delta(y)), \quad \forall\, y \in A(0,1/2,1).$$

Straightforward computations show that $\hat{f} \in L^2_0(A(0,1/2,1))$. Consequently, there exists $\hat{u} \in H^1_0(A(0,1/2,1))$ such that:

$$\begin{aligned}&\operatorname{div} \hat{u} = \hat{f} \text{ on } A(0,1/2,1) \\ &\|\nabla \hat{u}\|_{L^2(A(0,1/2,1))} \lesssim \|\hat{f}\|_{L^2(A(0,1/2,1))}.\end{aligned}$$

We set then:

$$u(x) = \left(\prod_{\ell\neq i} \sigma'_\delta(x_\ell)\hat{u}_i(Y_\delta(x))\right)_{i=1,2,3} \qquad \forall\, x \in A(0,1-1/\delta,1).$$

Since $\sigma'_\delta(x_\ell)\chi'_\delta(\sigma_\delta(x_\ell)) = 1$, we may expand the divergence to prove:

$$\operatorname{div} u(x) = f(x)\,, \quad \forall\, x \in A(0,1-1/\delta,1).$$

It is straightforward that $u = 0$ on the boundaries of $A(0,1-1/\delta,1)$, and we are left with computing the size of its gradient. We note that (introducing Kron the Kronecker symbol)

$$\partial_j u_i(x) = \sigma'_\delta(x_j)\left[\prod_{\ell\neq i}\sigma'_\delta(x_\ell)\right]\partial_j\hat{u}_i(Y_\delta(x)) + (1-\mathrm{Kron}[j,i])\sigma''_\delta(x_j)\left[\prod_{\ell\neq i,j}\sigma'_\delta(x_\ell)\right]\hat{u}_i(Y_\delta(x)).$$

Consequently:

$$\begin{aligned}\int_{A(0,1-1/\delta,1)} |\partial_j u_i(x)|^2 &\lesssim \int_{A(0,1-1/\delta,1)} \left(\delta^4 |\partial_j \hat{u}_i(Y_\delta(x))|^2 + \delta^9 |\hat{u}_i(Y_\delta(x))|^2\right) \prod_{\ell=1}^{3} \sigma'_\delta(x_\ell)\mathrm{d}x_\ell \\ &\lesssim \delta^9 \int_{A(0,1/2,1)} [|\partial_j \hat{u}(y)|^2 + |\hat{u}(y)|^2] \mathrm{d}y.\end{aligned}$$

Here we apply the classical Poincaré inequality in $H^1_0(A(0,1/2,1))$ and the definition of $\hat{u}$, which yields

$$\int_{A(0,1-1/\delta,1)} |\partial_j u_i(x)|^2 \lesssim \int_{A(0,1/2,1)} |\hat{f}(y)|^2 \mathrm{d}y.$$

We end up by dominating the right-hand side w.r.t. f recalling the bound above for χ'_δ :

$$\int_{A(0,1/2,1)} |\hat{f}(y)|^2 \mathrm{d}y = \int_{A(0,1/2,1)} \prod_{i=1}^{3} \chi'_\delta(y_i) |f(X_\delta(y_i))|^2 \prod_{i=1}^{3} \chi'_\delta(x_i) \mathrm{d}x_i \,,$$
$$\lesssim \int_{A(0,1-1/\delta,1)} |f(x)|^2 \mathrm{d}x.$$

□

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