DERIVATION OF THE STOKES-BRINKMAN PROBLEM AND EXTENSION TO THE DARCY REGIME.

M. HILLAIRET

Abstract. In this note, we consider the derivation of continuum models for the interactions between a cloud of particles and a viscous fluid. We review recent results on the derivation of so-called Stokes-Brinkman models. We give some insights into analytical tools that are required for such results and discuss a possible extension to the Darcy regime.

1. Introduction

In this note, we review results computing the asymptotics of the Stokes system in perforated domains when the number of holes in the domain goes to infinity while their radius goes to 0 with an appropriate scaling. Precisely, we fix Ω a smooth bounded domain of $\mathbb{R}^3$ and consider (u,p) the solution to:

$$(1) \qquad \begin{cases} -\Delta u + \nabla p = 0 \\ \mathrm{div} u = 0 \end{cases} \quad \text{in } \mathcal{F} := \Omega \setminus \overline{\bigcup_{i=1}^{N} B_i}.$$

Here the B_i represent the holes in Ω. We complement the problem with boundary conditions:

$$(2) \qquad \begin{cases} u = V_i & \text{on } \partial B_i \text{ for } i = 1,\dots,N, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $(V_1,\dots,V_N) \in [\mathbb{R}^3]^N$ are prescribed vectors. This problem can be seen as a model for computing the instantaneous response of a viscous fluid to the linear motion of a cloud of N immersed particles. For this reason, below we name B_i the "particles" and we assume the B_i lie in a bounded subdomain $K \subset \Omega$. However, we point out that the problem is stationary so that the "motion" of the particles is modeled here merely as non-zero boundary conditions.

When N is fixed and the B_i do not overlap nor intersect $\partial\Omega$, existence and uniqueness of a solution to system (1) completed with boundary conditions (2) is a standard issue. We recall here briefly the principle facts and refer the reader to [12] for more details. First, we can reduce (1)-(2) to a weak formulation satisfied by the velocity field u. Introducing the

Date: September 21, 2021.

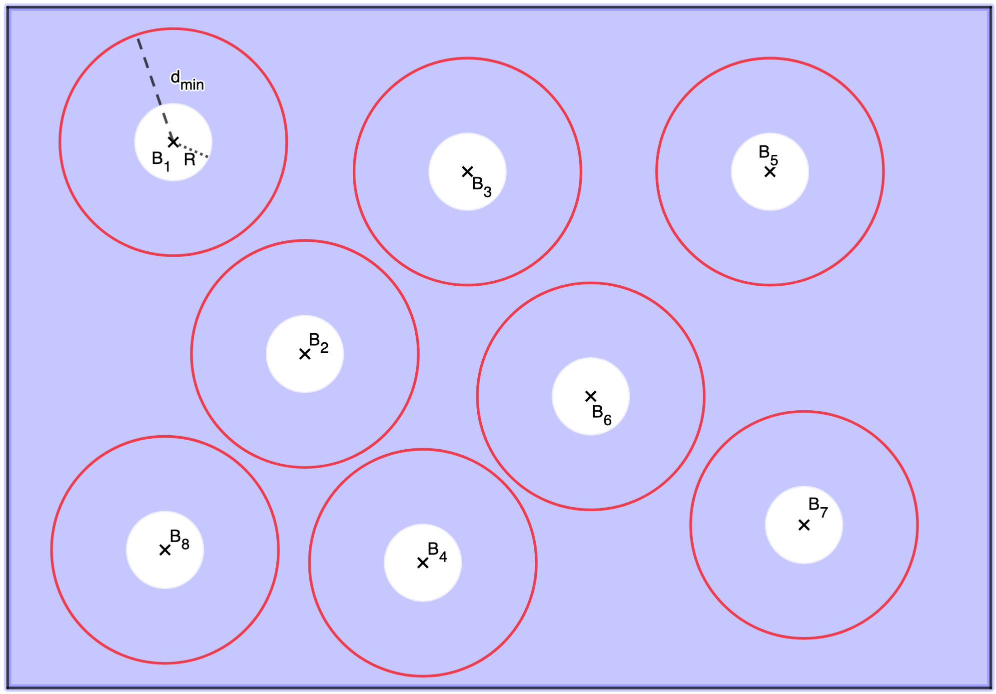

FIGURE 1. A 2D example of a typical configuration and notations : in blue the fluid domain, in white the particles, in red safety-spheres – centered in a particle and with radius $d_{min} \sim \lambda_{min} N^{-1/3}/2$– surrounding particles. For simplicity we draw euclidean safety-spheres.

function space $D(\mathcal{F})$ containing the restrictions of divergence-free $H^1_0(\Omega)$ vector fields to $\mathcal{F}$, this weak-formulation reads:

Find $u \in D(\mathcal{F})$ such that $u = V_i$ on ∂B_i for $i = 1, \dots, N$ and

$$\int_{\mathcal{F}} \nabla u : \nabla w = 0 \tag{3}$$

for all $w \in D(\mathcal{F})$ that vanish on $\partial\mathcal{F}$.

Standard arguments yield that there exists a unique such u and that it is characterized by minimizing:

$$\int_{\mathcal{F}} |\nabla u|^2$$

among the vector fields $w \in D(\mathcal{F})$ matching the same boundary conditions V_i on the ∂B_i. This property is actually general to any family of boundary conditions on the ∂B_i (possibly not constant but flux-free). The pressure p is then recovered as the Lagrange-multiplier of the divergence-free constraint. We tackle here the behavior of the velocity field u when N is large.

The results we detail below are highly sensitive to several geometrical parameters. We shall restrict herein to the simple case in which the particles are spherical and well-separated. Namely, we assume there exists $(h_1, \dots, h_N) \in [\mathbb{R}^3]^N$ and $R > 0$ so that $B_i = B(h_i, R)$. We ensure the particles are well-separated by assuming:

(H0) $h_i \in K$ for $i = 1, \dots, N$,
(H1) there exists $\lambda_{min} > 0$ such that

$$d_{min} := \min_{i=1,\dots,N} \{\min\{|h_i - h_j|_\infty, i \neq j\}, d_\infty(h_i, \partial\Omega)\} \geqslant \lambda_{min}/N^{1/3}.$$

We draw an example in Figure 1. We recall that K is the domain occupied by the cloud of particles. It is fixed a priori. The notations with index ∞ denote distances for the ℓ^∞ norm on $\mathbb{R}^3$. We keep notations $|\cdot|$ for the classical euclidean norm of vectors in $\mathbb{R}^3$ (and Lebesgue measure for sets). We denote by $\mathcal{C}$ the set of center families $\mathbf{h} := (h_1, \dots, h_N)$ satisfying assumptions (H0)-(H1). We note that this set contains families with an arbitrary large (but finite) number of centers and that it depends on the parameters K, λ_{min}. Since these parameters are fixed throughout the paper, we do not add them to specify the symbol $\mathcal{C}$ for legibility.

A geometry for solving (1)-(2) (which we call below "a configuration") is then completly determined by choosing a family of centers in $\mathcal{C}$ – which fixes implicitly the number of particles N – and a radius R. Since the value of R is restricted so that there is no overlap between the particles and between the particles and $\partial\Omega$, we shall see below R as a function of the chosen center family in $\mathcal{C}$. We will however call implicitly the choice of the mapping $\mathbf{h} \mapsto R$ by simply stating that "a radius R is chosen". Given a configuration, we fix a solution (u, p) to (1)-(2) by picking boundary data $(V_1, \dots, V_N)$. Again, we regroup the boundary data into a vector denoted by $\mathbf{V}$ whose length can be arbitrary large but matches the length of the vector $\mathbf{h}$.

In **Section 3** we assume the radius is chosen so that $R \leqslant r_0/N$ for some fixed parameter $r_0 > 0$. In this case, we introduce a description of the discrete set of particles and velocities *via* the functions:

$$\rho_d = R \sum_{i=1}^N \frac{1}{|\tilde{B}_i|} \mathbb{1}_{\tilde{B}_i}, \quad j_d = R \sum_{i=1}^N \frac{1}{|\tilde{B}_i|} \mathbb{1}_{\tilde{B}_i} V_i,$$

where $\tilde{B}_i = B_\infty(h_i, \lambda_{min} N^{-1/3}/2)$. When N is sufficiently large, the condition $R \leqslant r_0/N$ implies that $B_i \subset \tilde{B}_i$ for all i. Under this condition, we want also to replace these functions by continous descriptions of the cloud density $\rho : \Omega \to [0, \infty)$ and momentum $j : \Omega \to \mathbb{R}^3$ respectively. Indeed, it is expected (see [3]) that the solution (u, p) should be close to the solution (v, q) to a Stokes system with a relaxation term between the particle momentum and a virtual fluid momentum computed with respect to the particle density. This system, that we call Stokes-Brinkman, reads:

$$\begin{cases} -\Delta v + \nabla q = 6\pi(j - \rho v) \\ \mathrm{div} v = 0 \end{cases} \quad \text{in } \Omega, \tag{4}$$

with Dirichlet boundary condition:

$$v = 0 \quad \text{on } \partial\Omega. \tag{5}$$

In practice ρ and j should be continuous on K, it should also vanish outside K and could admit discontinuities on ∂K. We actually simply fix the "regularity" of ρ and j *via* integrability conditions. Indeed, under the assumption that the configuration is well-separated ($\mathbf{h} \in \mathcal{C}$ and $R \leqslant r_0/N$) we have an *a priori* bound above for $\|\rho_d\|_{L^\infty(\Omega)}$. Hence we restrict to the case $\rho \in L^\infty(\Omega)$. Similarly, we will typically assume a bound for boundary data $\mathbf{V}$ such as:

$$\|\mathbf{V}\|_{\ell^2}^2 := \frac{1}{N}\sum_{i=1}^{N} |V_i|^2 \leqslant M. \tag{6}$$

Correspondingly, we have a bound for j_d in $L^2(\Omega)$ and we restrict to the case $j \in L^2(\Omega)$.

Under the restriction $(\rho, j) \in L^\infty(\Omega) \times L^2(\Omega)$ with $\rho \geqslant 0$, existence and uniqueness of a solution to (4)-(5) is also standard. Like for the Stokes problem, we recall that (v, q) is constructed by introducing at first a weak-formulation satisfied by v. Introducing $D(\Omega)$ the set of divergence-free vector fields in $H^1_0(\Omega)$, this weak-formulation reads:

Find $v \in D(\Omega)$ such that:

$$\int_{\mathbb{R}^3} \nabla v : \nabla w + 6\pi \rho u \cdot w = 6\pi \int_{\mathbb{R}^3} j \cdot w, \tag{7}$$

for all $w \in D(\Omega)$.

Standard Lax-Milgram arguments yield existence and uniqueness of a v solution to this weak formulation. The pressure q is then recovered as the Lagrange-multiplier of the divergence-free constraint.

To compare v with u we extend u by the value V_i on B_i for $i = 1, \ldots, N$. We do not change the symbol for legibility. It is then clear that $u \in D(\Omega)$ and

$$\|\nabla u\|_{L^2(\Omega)} = \|\nabla u\|_{L^2(\mathcal{F})}.$$

With these conventions, our main result reads:

Theorem 1. *There exists a constant $C(\Omega, \lambda_{min}, r_0)$ depending only on $\Omega, \lambda_{min}, r_0$ such that, if N is sufficiently large:*

$$\begin{aligned} \|u - v\|_{L^2(\Omega)} \leqslant C(\Omega, \lambda_{min}, r_0)(\|\rho\|_{L^\infty(\Omega)} + 1)\ldots \\ \ldots\left(\frac{\|\mathbf{V}\|_{\ell^2}}{N^{1/6}} + \|j - j_d\|_{(H^2(\Omega)\cap H^1_0(\Omega))^*} + \|\rho - \rho_d\|_{H^{-1}(\Omega)}\|\mathbf{V}\|_{\ell^2}\right). \end{aligned} \tag{8}$$

We propose an extensive proof in **Section 3** relying on a good interpretation of the weak formulation satisfied by u. This novel interpretation is presented in **Section 2**. Our proof of **Theorem 1** is directly adapted from [12, 14, 16].

The content of **Theorem 1** enables to address the limit of the sequence of weak solutions $u^{(N)}$ to (1)-(5) associated with N-particle configurations $\mathbf{h}^N$ in $\mathcal{C}$, radius $R^{(N)} = r_0/N$

and boundary data $\mathbf{V}^{(N)}$ satisfying uniformly (6). In this framework, the inequality (8) yields a quantified version of the convergence of $u^{(N)}$ toward the weak solution v to the Stokes-Brinkman problem with corresponding density and momentum. In the case of well-separated configurations that we consider here, it extends [5] which adapts previous results in [1] for the homogeneous case. In [12], we consider an analogous convergence problem for more general configurations for which the distance between two particles is infinitly larger than their radius and there is no concentration of the particles in subsets. This other result enables to tackle a random distribution of particles, see [4]. Similar computations are also provided for particles with arbitrary shapes in [14]. The quantified version of the convergence that we provide here is inspired of [16]. Analogous results are also obtained in the case $\Omega = \mathbb{R}^3$ by using a specific representation of the solution via a "method of reflections" [15]. These convergence results in the Brinkman regime are also complemented and related to Einstein's problem for the computation of effective viscosity in [6].

In [5], the authors mention in the introduction that the condition (6) is the good scaling for boundary data $\mathbf{V}$ to yield the Stokes-Brinkman problem because the radius of particles scales like $1/N$. However, the situation where we choose a larger radius R is left open. In particular, how to scale boundary data to extend the results in the Darcy regime of [2] (which is obtained in case $\mathbf{V}$ vanishes and the system is forced by a volumic force) is left open. This is the question that we tackle in the following.

In **Section 4,** we show that the new interpretation of the weak-formulation satisfied by u in **Section 2** enables to tackle the case of arbitrary large choices for R. Namely, we assume the choice of radius satisfies:

$$\frac{r_0}{N} \leqslant R \leqslant \frac{\lambda_{min}}{4N^{1/3}}, \tag{9}$$

for some $r_0 > 0$ and when N is sufficiently large so that such a condition is not contradictory. We note that, with this choice, there still holds that $B_i \subset \tilde{B}_i$ for all i. To state our result, we fix a target velocity field $v : \Omega \to \mathbb{R}^3$ for the fluid such that $v \in L^2(\Omega)$. Correspondingly, we introduce functions encoding the discrete set of velocities V_i and particles. Precisely, we set:

$$\tilde{v}_d(x) = \sum_{i=1}^{N} V_i \mathbb{1}_{\tilde{B}_i}, \quad \tilde{\rho}_d(x) = \sum_{i=1}^{N} \mathbb{1}_{\tilde{B}_i}. \tag{10}$$

We note that, $\tilde{\rho} \in L^\infty(\Omega)$ and that, since the $\tilde{B}_i$ are disjoint, there holds:

$$\|\tilde{v}_d\|_{L^2(\Omega)} \leqslant C(\lambda_{min})\|\mathbf{V}\|_{\ell^2}.$$

We obtain the following result:

Theorem 2. *With the assumption* (9) *and the construction* (10)*, let denote* $\tilde{u} = \sqrt{\tilde{\rho}_d} u$. *Then, there exists a constant* $C(\Omega, \lambda_{min}, r_0)$ *depending only on* $\Omega, \lambda_{min}, r_0$ *such that:*

$$\|v - \tilde{u}\|_{[H^3(\Omega)\cap D(\Omega)]^*} \leqslant C(\Omega, \lambda_{min}, r_0) \left(\|v - \tilde{v}_d\|_{[H^3(\Omega)\cap D(\Omega)]^*} + \frac{\sqrt{RN}}{N^{1/3}} + \frac{1}{\sqrt{RN}} \right) \|\mathbf{V}\|_{\ell^2}.$$

The inequality of **Theorem 2** means simply that, for arbitrary $w \in H^3(\Omega) \cap D(\Omega)$:

$$\left| \int_\Omega (v - \tilde{u}) \cdot w \right| \leqslant C(\Omega, \lambda_{min}, r_0) \left(\|v - \tilde{v}_d\|_{[H^3(\Omega)\cap D(\Omega)]^*} + \frac{\sqrt{RN}}{N^{1/3}} + \frac{1}{\sqrt{RN}} \right) \|\mathbf{V}\|_{\ell^2} \|w\|_{H^3(\Omega)}.$$

Several remarks are in order. First, we point out that $\tilde{\rho}_d$ is an indicator function so that $\sqrt{\tilde{\rho}_d} = \tilde{\rho}_d$. We can then define indifferently $\tilde{u} = \tilde{\rho}_d u$. We note also that, in case the particles are distributed periodically in a cube K, and λ_{min} is chosen as large as possible, we have that $\tilde{\rho}_d = \mathbb{1}_K + rem$ with $\|rem\|_{L^1(\Omega)} \leqslant 1/N^{1/3}$. We may even have $rem = 0$ if the particles are well-centered in K. In this periodic well-centered case, we can thus interpret $\tilde{u}$ as the restriction of u to K and our result extends to the case of non-vanishing boundary conditions the content of [2, Section 3] (see also more recently [11] when particles are distributed randomly). Going through the proof, it is possible to gain a little on the dual space in which we measure our distance. A priori, we use mostly that $w \in W^{1,6}(\Omega) \cap C^{0,1}(\bar{\Omega})$. We consider the space $H^3(\Omega)$ to ensure this property in a Hilbert setting. We point out that we test on divergence-free functions w so that the result of **Theorem 2** gives an information on $v - \tilde{u}$ up to the addition of the gradient of a pressure only. This has to be expected since we might have $\tilde{u} = u$ in which case it is divergence-free, while v (or $\tilde{v}_d$) need not be.

Like for the Brinkman regime, the content of **Theorem 2** enables to address the limit of the sequence of weak solutions $u^{(N)}$ to (1)-(5) associated with N-particle configurations $\mathbf{h}^{(N)}$ in $\mathcal{C}$ and boundary data $\mathbf{V}^{(N)}$ satisfying uniformly (6) in case the radius $R^{(N)}$ satisfy:

$$R^{(N)} = \frac{r_{sc}}{N^\alpha}$$

for some $r_{sc} > 0$, and $\alpha \in [1/3, 1]$. When $\alpha = 1$, we see that we do not obtain a better estimate than an error term of order $1/\sqrt{r_{sc}}$. This could be expected since we do know already with **Theorem 1** a more precise approximation to $u^{(N)}$. When $\alpha \in (1/3, 1)$ the remainder vanishes when N diverges. Our result then implies that the velocity-field prescribed by the particles spreads in the fluid. When $\alpha = 1/3$, the situation is more involved. The remainder does not go to 0 when N diverges but it can be made as small as desired choosing r_{sc} sufficiently small and N sufficiently large. For instance, we can apply our computations to the case where the data $\mathbf{V}$ are fixed implicitly by requiring that the forces on the particles vanish. This kind of boundary conditions is proposed by Einstein in order to compute the effective viscosity of a mixture fluid+particles . Thus, this direction should yield an alternative derivation of the effective vicosity formula (see [13, 7] and also [8, 9, 10] and references therein). But the way the implicit computations of the V_i interacts with the limit $N \to \infty$ has still to be explored.

To end up this introductory section, we give below a list of main notations and recall that we use mainly the euclidean norm $|\cdot|$ on $\mathbb{R}^3$ and the ℓ^∞ distance that we identify with the subscript ∞. In particular, the distance d_∞ and the balls B_∞ are constructed wrt the ℓ^∞ distance. Finally, like in our main results, we use throughout the paper notations C for generic constants. These constants may vary between lines. If necessary, we point out the important dependencies in parenthesis.

$\Omega, \mathcal{F}, K$:	container, fluid domain, cloud,
$B_i, \tilde{B}_i$:	particle i, safety-cube surrounding particle i,
N, R, h_i :	number of particles, radius of particles, center of particle i,
$V_i, \mathbf{V}$:	boundary condition on particle i, list of boundary conditions,
$\rho, \rho_d, \tilde{\rho}_d$:	density of the cloud of particles and discrete counterparts,
j, j_d :	particle momentum and discrete counterpart.
$D(\Omega)$:	space of $H^1_0(\Omega)$ divergence-free vector fields
$D(\mathcal{F})$:	restriction of vector fields in $D(\Omega)$ to $\mathcal{F}$.

Table 1. Main notations

The paper is organized as follows. The next section is devoted to a key technical lemma that is central to our main results. **Section 3** is then devoted to the Brinkman regime in which $R \leqslant r_0/N$ for some constant $r_0 > 0$. In this case we recover **Theorem 1**. This part is mostly a rewriting of previous approaches on the topic. The last section is devoted to the Darcy regime in which $R \geqslant r_0/N$ for some constant $r_0 > 0$ and we detail the proof of **Theorem 2**. This part is completly new to our knowledge and extend previous computations in the Darcy regime to the case of a moving porous medium (the particles in our case).

Acknowledgement. The author would like to thank D. Gérard-Varet, R. M. Höfer and the anonymous referee for many suggestions to improve previous versions of the paper.

2. A key preliminary lemma

In this first section, we prove a technical result that is at the heart of the following computations. This result applies to general configurations since we shall only assume that $\mathbf{h} \in \mathcal{C}$ and that R is chosen so that:

$$B(h_i, 2R) \subset B(h_i, \lambda_{min} N^{-1/3}/4) \quad \forall\, i = 1, \dots, N. \tag{11}$$

The main result of this section reads then:

Lemma 3. *Assume that* (11) *holds true and denote by u the unique weak solution to* (1)-(2) *associated with data* $\mathbf{V}$. *Then, there exists a linear mapping* $err \in [H^2(\Omega) \cap H^1_0(\Omega)]^*$ *satisfying:*

i) *the bound:*

$$\|err\|_{[H^2(\Omega)\cap H^1_0(\Omega)]^*} \leqslant C(\Omega,\lambda_{min}) R N^{2/3} \|\nabla u\|_{L^2(\Omega)},$$

ii) *for any* $w \in H^2(\Omega)\cap D(\Omega)$, *setting:*

$$\bar{w}_i = \frac{1}{|B_i|}\int_{B_i} w(x) \quad \text{and} \quad \tilde{u}_i = \frac{1}{|\tilde{B}_i|}\int_{\tilde{B}_i} u(x), \quad \forall\, i = 1,\ldots,N,$$

there holds:

$$\int_\Omega \nabla u : \nabla w = 6\pi R \sum_{i=1}^N \bar{w}_i \cdot (V_i - \tilde{u}_i) + \langle err, w\rangle. \tag{12}$$

What remains of this section is devoted to the proof of this lemma. For this, we need the solution $(\mathcal{U}_R[V],\mathcal{P}_R[V])$ to the Stokes equations outside one particle of radius R in $\mathbb{R}^3$:

$$\mathcal{U}_R[V](x) = \nabla \times \left[\left(\frac{3R}{2|x|} - \frac{R^3}{2|x|^3}\right)\frac{V\times x}{2}\right] \quad \text{and} \quad \mathcal{P}_R[V](x) = \frac{3RV}{2}\cdot\nabla\left(\frac{1}{|x|}\right).$$

We mean that $(\mathcal{U}_R,\mathcal{P}_R)$ is a solution to

$$\begin{cases} -\Delta\mathcal{U}_R[V] + \nabla\mathcal{P}_R[V] = 0 & \text{in } \mathbb{R}^3\setminus B(0,R) \\ \operatorname{div}\mathcal{U}_R[V] = 0 & \text{in } \mathbb{R}^3\setminus B(0,R) \end{cases}$$

with boundary conditions:

$$\mathcal{U}_R[V](x) = V \text{ on } \partial B(0,R) \quad \text{and} \quad \lim_{|x|\to\infty}\mathcal{U}_R[V](x) = 0.$$

We also recall one particular algebraic property (known as Stokes formula for the drag):

$$\int_{\partial B(0,R)} \partial_n\mathcal{U}_R[V](x) - \mathcal{P}_R[V](x) n \mathrm{d}\sigma = 6\pi R V, \tag{13}$$

where n stands for the normal to $\partial B(0,R)$ directed towards the interior of $B(0,R)$. We keep this convention for normal vectors to spheres in what follows. Usually, this formula is written with the symmetric-gradient of $\mathcal{U}_R$. But, both formulas coincide because $\mathcal{U}_R[V]$ is divergence-free and constant on $\partial B(0,R)$. As a straightforward consequence, we mention that, since the Stokes equations state the conservation of normal stress we have also that:

$$\int_{\partial B_\infty(0,\tilde{R})} \partial_n\mathcal{U}_R[V](x) - \mathcal{P}_R[V](x) n \mathrm{d}\sigma = 6\pi R V, \qquad \forall\, \tilde{R} \geqslant R. \tag{14}$$

We need below also a truncated variant of $\mathcal{U}_R$, namely:

$$\mathcal{U}_R^N[V](x) = \nabla\times\left[\chi\left(4N^{1/3}|x|/\lambda_{min}\right)\left(\frac{3R}{2|x|} - \frac{R^3}{2|x|^3}\right)\frac{V\times x}{2}\right],$$

where χ is a smooth truncation function satisfying $\mathbb{1}_{(-1,1)} \leqslant \chi \leqslant \mathbb{1}_{(-2,2)}$. We remark that $\mathcal{U}_R^N[V]$ is divergence-free and satisfies:

$$\mathcal{U}_R^N[V] = \begin{cases} \mathcal{U}_R[V] & \text{on } B(0,\lambda_{min}N^{-1/3}/4), \\ 0 & \text{outside } B(0,\lambda_{min}N^{-1/3}/2). \end{cases}$$

We fix now $w \in H^2(\Omega) \cap D(\Omega)$ and compute the left-hand side of (12) by using the Stokes system satisfied by u. Indeed, noting that $\nabla u = 0$ on B_i, that div $w = 0$ on Ω and integrating by parts, we have:

$$\begin{aligned}\int_{\Omega} \nabla u : \nabla w &= \int_{\Omega \setminus \bigcup B_i} \nabla u : \nabla w \\ &= \sum_{i=1}^{N} \int_{\partial B_i} (\partial_n u - pn) \cdot w \mathrm{d}\sigma =: \sum_{i=1}^{N} L_i[w].\end{aligned}$$

Given $i \in \{1, \ldots, N\}$, we introduce now the means $\bar{w}_i$ and $\tilde{u}_i$ that are defined in the item ii) of **Lemma 3** and split the integral $L_i[w]$ into:

$$\begin{aligned}L_i[w] &= \int_{\partial B_i} (\partial_n u - pn) \cdot \mathcal{U}_R^N[\bar{w}_i](\cdot - h_i) \mathrm{d}\sigma + err_i^1 \\ &= \int_{\tilde{B}_i} \nabla u : \nabla \mathcal{U}_R^N[\bar{w}_i](x - h_i) + err_i^1 \\ &= \int_{\tilde{B}_i} \nabla u : \nabla \mathcal{U}_R[\bar{w}_i](x - h_i) + err_i^2 + err_i^1 \\ &= 6\pi R \bar{w}_i \cdot (V_i - \tilde{u}_i) - err_i^3 + err_i^2 + err_i^1.\end{aligned}$$

We point out that the $6\pi R$ term comes from the formulas (13)-(14), while the error terms come from truncation and reduction to the constant values $\bar{w}_i, \tilde{u}_i$. Precisely, we have:

$$\begin{aligned}err_i^1 &:= \int_{\partial B_i} (\partial_n u - pn) \cdot (w - \bar{w}_i) \mathrm{d}\sigma \\ err_i^2 &:= \int_{\tilde{B}_i} \nabla u : \nabla (\mathcal{U}_R^N[\bar{w}_i](x - h_i) - \mathcal{U}_R[\bar{w}_i](x - h_i)) \\ err_i^3 &:= \int_{\partial \tilde{B}_i} (\partial_n \mathcal{U}_R[\bar{w}_i](x - h_i) - \mathcal{P}_R[\bar{w}_i](x - h_i) n) \cdot (u - \tilde{u}_i) \mathrm{d}\sigma.\end{aligned}$$

So, we define:

$$\langle err, w \rangle = \sum_{i=1}^{N} \sum_{k=1}^{3} err_i^k.$$

With such a definition, we have that err is indeed a linear form on $H^2(\Omega) \cap D(\Omega)$ satisfying item ii). We only need to show that err is bounded with the bound of item i) to complete the proof. This is the content of the next computations.

Proof of item i). We control the three error terms:

$$\sum_{i=1}^{N} err_i^k$$

for $k = 1, 2, 3$, independently.

In the case $k=1$, for a given $i \in \{1,\dots,N\}$, we construct

$$\hat{w}_i(x) = \chi\left(\frac{|x-h_i|}{R}\right)(w(x)-\bar{w}_i) + b_i(x)$$

where χ is the same truncation function as previously and $b_i \in H^1_0(B(h_i,2R)\setminus B(h_i,R))$ is introduced to compensate the divergence of the first term. Via standard scaling argument and Poincaré-Wirtinger inequality (see the appendix of [14]), we obtain that:

$$|err^1_i| = \left|\int_{B(h_i,2R)\setminus B(h_i,R)} \nabla u : \nabla \hat{w}_i\right| \leqslant C\|\nabla u\|_{L^2(B(h_i,2R))}\|\nabla w\|_{L^2(B(h_i,2R))}.$$

Summing over i and applying standard Hölder and Cauchy-Schwarz inequalities with the embedding $H^2(\Omega)\subset W^{1,6}(\Omega)$ we conclude that

$$\sum_{i=1}^N |err^1_i| \leqslant \|\nabla u\|_{L^2(\Omega)}\left|\bigcup_{i=1}^N B(h_i,2R)\right|^{\frac13}\|w\|_{H^2(\Omega)} \leqslant C(\Omega)RN^{1/3}\|\nabla u\|_{L^2(\Omega)}\|w\|_{H^2(\Omega)}.$$

In the case $k=2$, we use the decay properties of $\mathcal{U}_R$:

$$|\mathcal{U}_R[V](x)| + |x||\nabla\mathcal{U}_R[V](x)| \leqslant \frac{CR|V|}{|x|} \quad \forall\,|x|\geqslant R, \quad \forall\, V\in\mathbb{R}^3,$$

that entail:

$$|\nabla\mathcal{U}^N_R[V](x)| \leqslant \frac{CR|V|}{|x|^2} \quad \forall\,|x|\geqslant R, \quad \forall\, V\in\mathbb{R}^3.$$

Consequently, for $i=1,\dots,N$, applying that $\mathcal{U}^N_R[\bar{w}_i]-\mathcal{U}_R[\bar{w}_i]$ vanishes at a distance lower than $\lambda_{min}N^{-1/3}/4$ from the center, we obtain:

$$\begin{aligned}
|err^2_i| &\leqslant \|\nabla u\|_{L^2(\tilde{B}_i)}\left(\|\nabla\mathcal{U}^N_R[\bar{w}_i]\|_{L^2(\mathbb{R}^3\setminus B(0,\lambda_{min}N^{-1/3}/4))} + \|\nabla\mathcal{U}_R[\bar{w}_i]\|_{L^2(\mathbb{R}^3\setminus B(0,\lambda_{min}N^{-1/3}/4))}\right)\\
&\leqslant \|\nabla u\|_{L^2(\tilde{B}_i)}\left(\int_{\lambda_{min}N^{-1/3}/4}^{\infty}\frac{|R|^2|\bar{w}_i|^2}{r^2}\mathrm{d}r\right)^{\frac12}\\
&\leqslant \|\nabla u\|_{L^2(\tilde{B}_i)}C(\lambda_{min})RN^{1/6}|\bar{w}_i|.
\end{aligned}$$

Consequently with a standard Cauchy-Schwarz inequality and applying the embedding $H^2(\Omega)\subset L^\infty(\Omega)$ we obtain:

$$\sum_{i=1}^N |err^2_i| \leqslant C(\lambda_{min})\|\nabla u\|_{L^2(\Omega)}RN^{2/3}\|w\|_{H^2(\Omega)}$$

For the last case $k=3$, we proceed as for the first error term by lifting the boundary error $u-\tilde{u}_i$. We introduce an alternative truncation function $\zeta:\mathbb{R}\to[0,1]$ such that $\zeta=1$ on $(0,\infty)$ and $\zeta=0$ on $(-\infty,-1/2)$. For fixed i we set then:

$$\hat{u}_i = \zeta\left(\frac{2N^{\frac13}}{\lambda_{min}}|x-h_i|_\infty - 1\right)(u-\tilde{u}_i) - \beta_i,$$

where $\beta_i \in H^1_0(B_\infty(h_i,\lambda_{min}N^{-1/3}/2) \setminus B_\infty(h_i,\lambda_{min}N^{-1/3}/4))$ lifts the divergence induced by the previous truncation. Again via scaling arguments and a Poincaré-Wirtinger type inequality (see the appendix of [14]) we have:

$$\begin{aligned}|err^3_i| &= \left|\int_{B_\infty(h_i,\lambda_{min}N^{-1/3}/2)\setminus B_\infty(h_i,\lambda_{min}N^{-1/3}/4)} \nabla\hat{u}_i : (\nabla\mathcal{U}_R[\bar{w}_i](x-h_i) - \mathcal{P}_R[\bar{w}_i](x-h_i))\right|\\ &\leqslant C(\lambda_{min})\|\nabla u\|_{L^2(B_\infty(h_i,\lambda_{min}N^{-1/3}/2))}\cdots\\ &\quad ...\|(\nabla\mathcal{U}_R[\bar{w}_i](x-h_i) - \mathcal{P}_R[\bar{w}_i](x-h_i))\|_{L^2(\mathbb{R}^3\setminus B(h_i,\lambda_{min}N^{-1/3}/4))))}.\end{aligned}$$

Via similar computations as for the second error term, we obtain again that:

$$\sum_{i=1}^N |err^3_i| \leqslant C(\lambda_{min})RN^{2/3}\|\nabla u\|_{L^2(\Omega)}.$$

This concludes the proof. □

3. The Brinkman regime

In this section, we focus on the Brinkman regime in which we choose a radius $R \leqslant r_0/N$ for some $r_0 > 0$. We do not require any smallness on the parameter r_0. But, we assume N sufficiently large so that condition (H1) ensures (11). This condition does not restrict the generality *a priori* since we are interested in representing the discrete set of particles by a continuous medium which requires a sufficiently large amount of particles. We detail here the proof of **Theorem 1** and its consequences.

3.1. Proof of Theorem 1.

So, we want to compare u with the velocity field v of the solution (v,q) to the Stokes-Brinkman problem (4)-(5) associated with the pair (ρ,j) that we have set in the introduction. We recall that $v \in D(\Omega)$ is characterized by the set of equations:

$$\int_\Omega \nabla v : \nabla w + 6\pi\int_\Omega \rho v\cdot w = 6\pi\int_\Omega j\cdot w\,, \quad \forall\, w \in D(\Omega). \tag{15}$$

Our proof goes into two steps. Firstly, we show a bound for u in $H^1_0(\Omega)$ in terms of:

$$\|\mathbf{V}\|_{\ell^2} := \left(\frac{1}{N}\sum_{i=1}^N |V_i|^2\right)^{\frac12}.$$

This is the content of the following lemma:

Lemma 4. *There exists a constant $C(r_0)$ such that,*

$$\|\nabla u\|_{L^2(\Omega)} \leqslant C(r_0)\|\mathbf{V}\|_{\ell^2}.$$

Proof. We apply the minimization principle satisfied by solutions to the Stokes system: we have that $\|\nabla u\|_{L^2(\mathcal{F})}$ is minimal among the vector fields $w \in D(\mathcal{F})$ that match the boundary condition $w = V_i$ on ∂B_i. So, we construct a divergence-free lifting u_0 of these boundary conditions. This can be done by a suitable truncation of the constant vector field

V_i, by lifting then the divergence of the induced truncation and argue by scaling. However, for such simple boundary data, we propose to construct explicitly the lifting. Namely, since we have chosen N sufficiently large so that the no-overlap condition (11) holds true, we set:

$$u_0(x) = \sum_{i=1}^{N} \nabla \times \left[\chi\left(\frac{|x-h_i|}{R}\right) \frac{V_i \times (x-x_i)}{2} \right]. \tag{16}$$

where χ is ever the same smooth truncation function satisfying $\mathbb{1}_{(-1,1)} \leqslant \chi \leqslant \mathbb{1}_{(-2,2)}$. Due to condition (11), the vector fields in the above sum have pairwise disjoint supports. Furthermore, they are by construction divergence-free and satisfy:

$$\nabla \times \left[\chi\left(\frac{|x-h_i|}{R}\right) \frac{V_i \times (x-x_i)}{2} \right] = \nabla \times \left[\frac{V_i \times (x-x_i)}{2} \right] = V_i \qquad \text{on } \partial B_i.$$

We complete the proof by remarking that

$$\|\nabla u_0\|_{L^2(\mathcal{F})} \leqslant C\sqrt{RN}\|\mathbf{V}\|_{\ell^2}.$$

□

We can now involve the computations of **Lemma 3** to use a standard characterization of the $L^2(\Omega)$-norm (inside divergence-free $H^1_0(\Omega)$ vector fields):

$$\|u-v\|_{L^2(\Omega)} = \sup\left\{ \int_{\Omega} (u-v)\cdot\varpi \quad \varpi \in D(\Omega), \quad \|\varpi\|_{L^2(\Omega)} = 1 \right\}. \tag{17}$$

So, we fix a divergence-free $\varpi \in D(\Omega)$ with unit L^2-norm and introduce $w \in D(\Omega)$ the velocity field of the unique (weak) solution to the Stokes-Brinkman problem:

$$\begin{cases} -\Delta w + 6\pi\rho w + \nabla q = \varpi \\ \qquad\qquad \operatorname{div} w = 0 \end{cases} \quad \text{in } \Omega$$

with boundary conditions

$$w = 0 \quad \text{on } \partial\Omega.$$

We note that, since $\rho \in L^\infty(\Omega)$ and $\varpi \in L^2(\Omega)$, standard ellipticity arguments entail that $w \in H^2(\Omega)$ with:

$$\|w\|_{H^1_0(\Omega)} + \|w\|_{H^2(\Omega)} \leqslant C(\Omega)(\|\rho\|_{L^\infty(\Omega)} + 1). \tag{18}$$

We have then by integration by parts:

$$\begin{aligned} \int_{\Omega} (u-v)\cdot\varpi &= \int_{\Omega} (u-v)\cdot(-\Delta w + 6\pi\rho w + \nabla q) \\ &= \int_{\Omega} \nabla(u-v) : \nabla w + \int_{\Omega} 6\pi\rho w \cdot (u-v). \end{aligned}$$

In the right-hand side, we extract the v term and apply (15). As for the u term, we apply **Lemma 3** and, keeping notations, we rewrite:

$$\int_\Omega \nabla u : \nabla w + \int_\Omega 6\pi \rho w \cdot u = 6\pi R \sum_{i=1}^N \bar{w}_i \cdot (V_i - \tilde{u}_i) + \langle err, w \rangle + \int_\Omega 6\pi\rho w \cdot u.$$

Regrouping both computations, we obtain finally that:

$$\int_\Omega (u-v) \cdot \varpi = 6\pi(\langle T_h, w\rangle - \langle T_f, w \rangle) + \langle err, w\rangle,$$

where

$$\langle T_f, w \rangle = \int_\Omega j \cdot w - \sum_{i=1}^N R V_i \cdot \bar{w}_i\,, \qquad \text{and} \qquad \langle T_h, w\rangle = \int_\Omega \rho u \cdot w - \sum_{i=1}^N R\tilde{u}_i \cdot \bar{w}_i.$$

Combining **Lemma 3** with **Lemma 4** and (18), we have that:

$$|\langle err, w\rangle| \leqslant \frac{C(\Omega, \lambda_{min}, r_0)}{N^{1/3}}(\|\rho\|_{L^\infty(\Omega)} + 1)\|\mathbf{V}\|_{\ell^2}.$$

So, we conclude the proof of **Theorem 1** by showing the following lemma:

Lemma 5. *There exists a constant $C(\Omega, \lambda_{min}, r_0)$ such that:*

$$|\langle T_f, w\rangle| \leqslant C(\Omega, \lambda_{min}, r_0)(\|\rho\|_{L^\infty(\Omega)} + 1)\left(\|j - j_d\|_{(H^2(\Omega)\cap H^1_0(\Omega))^*} + \frac{\|\mathbf{V}\|_{\ell^2}}{N^{1/6}}\right),$$

$$|\langle T_h, w\rangle| \leqslant C(\Omega, \lambda_{min}, r_0)(\|\rho\|_{L^\infty(\Omega)} + 1)\left(\|\rho - \rho_d\|_{H^{-1}(\Omega)}\|\mathbf{V}\|_{\ell^2} + \frac{\|\mathbf{V}\|_{\ell^2}}{N^{1/6}}\right),$$

where we recall that:

$$\rho_d = R\sum_{i=1}^N \frac{1}{|\tilde{B}_i|}\mathbb{1}_{\tilde{B}_i} \quad j_d = R\sum_{i=1}^N \frac{1}{|\tilde{B}_i|}\mathbb{1}_{\tilde{B}_i} V_i. \tag{19}$$

Proof. For the first term, we first remark that $w \in H^2(\Omega) \subset C^{0,1/2}(\bar{\Omega})$. This entails that, for any $i \in \{1, \dots, N\}$

$$\left|\frac{1}{|B_i|}\int_{B_i} w(x) - \frac{1}{|\tilde{B}_i|}\int_{\tilde{B}_i} w(x)\right| \leqslant \frac{C(\lambda_{min})\|w\|_{H^2(\Omega)}}{N^{1/6}}.$$

Introducing (18) and the definition (19) of j_d, we derive then:

$$\left|\sum_{i=1}^N RV_i \cdot \bar{w}_i - \int_\Omega j_d(x) \cdot w(x)\right| \leqslant \frac{C(\Omega, \lambda_{min}, r_0)}{N^{1/6}}\|\mathbf{V}\|_{\ell^2}\left(\|\rho\|_{L^\infty(\Omega)} + 1\right)$$

This entails:

$$\begin{aligned}|\langle T_f, w\rangle| &\leqslant |\langle j - j_d, w\rangle| + \frac{C(\Omega,\lambda_{min},r_0)}{N^{1/6}}\|\mathbf{V}\|_{\ell^2}\left(\|\rho\|_{L^\infty(\Omega)}+1\right)\\ &\leqslant \|j-j_d\|_{(H^2(\Omega)\cap H^1_0(\Omega))^*}\|w\|_{H^2(\Omega)} + \frac{C(\Omega,\lambda_{min},r_0)}{N^{1/6}}\|\mathbf{V}\|_{\ell^2}\left(\|\rho\|_{L^\infty(\Omega)}+1\right)\\ &\leqslant C(\Omega,\lambda_{min},r_0)(\|\rho\|_{L^\infty(\Omega)}+1)\left(\|j-j_d\|_{(H^2(\Omega)\cap H^1_0(\Omega))^*} + \frac{\|\mathbf{V}\|_{\ell^2}}{N^{1/6}}\right).\end{aligned}$$

As for the other term, we first rephrase:

$$\begin{aligned}\sum_{i=1}^N R\tilde{u}_i\cdot\bar{w}_i &= R\sum_{i=1}^N \frac{1}{|\tilde{B}_i|}\int_{\tilde{B}_i} u\cdot\bar{w}_i\\ &= \int_\Omega \rho_d(x)u(x)\cdot w(x) + R\sum_{i=1}^N\frac{1}{|\tilde{B}_i|}\int_{\tilde{B}_i} u(x)\cdot(w(x)-\bar{w}_i).\end{aligned}$$

Here we apply that $w\in H^2(\Omega)\subset C^{0,1/2}(\bar\Omega)$ which entails that:

$$|w-\bar{w}_i|\leqslant (diam(\tilde{B}_i))^{\frac12}\|w\|_{H^2(\Omega)} \quad \text{ on } \tilde{B}_i.$$

Consequently, applying again the embedding $H^1_0(\Omega)\subset L^6(\Omega)$ and the bound (18) we get:

$$\begin{aligned}\left|R\sum_{i=1}^N\frac{1}{|\tilde{B}_i|}\int_{\tilde{B}_i} u\cdot(w-\bar{w}_i)\right| &\leqslant \sum_{i=1}^N \frac{r_0}{N|\tilde{B}_i|^{\frac16}}\|u\|_{L^6(\tilde{B}_i)}\max_i\sup_{\tilde{B}_i}|w-\bar{w}_i|\\ &\leqslant \frac{C(\Omega,\lambda_{min},r_0)}{N^{1/6}}(\|\rho\|_{L^\infty(\Omega)}+1)\|\nabla u\|_{L^2(\Omega)}.\end{aligned}$$

Splitting $\rho=\rho-\rho_d+\rho_d$ as we did for the momentum, we obtain finally:

$$|\langle T_h, w\rangle|\leqslant C(\Omega,\lambda_{min},r_0)(\|\rho\|_{L^\infty(\Omega)}+1)\left(\|\rho-\rho_d\|_{H^{-1}(\Omega)}\|\mathbf{V}\|_{\ell^2}+\frac{\|\mathbf{V}\|_{\ell^2}}{N^{1/6}}\right).$$

□

3.2. **Comments.** The previous computations naturally applies to tackling the issue $N\to\infty$ in the system (1)-(2). Precisely, assume we choose now as parameter the number N of particles in Ω and consider a sequence of configurations $(\mathbf{h}^{(N)}, R^{(N)})$ where $\mathbf{h}^{(N)}\in\mathcal{C}$ contains N centers and $R^{(N)}\leqslant r_0/N$ uniformly for some constant $r_0>0$. We also consider a sequence $\mathbf{V}^{(N)}$ of boundary data. To this sequence of boundary data, we associate the discrete momentums and densities:

$$\rho_d^{(N)} := R^{(N)}\sum_{i=1}^N\frac{1}{|\tilde{B}_i^{(N)}|}\mathbb{1}_{\tilde{B}_i^{(N)}}, \qquad j_d^{(N)} := R^{(N)}\sum_{i=1}^N\frac{1}{|\tilde{B}_i^{(N)}|}\mathbb{1}_{\tilde{B}_i^{(N)}}V_i^{(N)},$$

with obvious notations. Assume further that

(H2) there exists a constant M such that $\|\mathbf{V}^{(N)}\|_{\ell^2}\leqslant M$ for all N.

With (H2), we note that $\rho_d^{(N)}$ is bounded in $L^\infty(\Omega)$ and $j_d^{(N)}$ is bounded in $L^2(\Omega)$ and that both have support in K. We can then assume that they converge respectively to some $\rho \in L^\infty(\Omega)$ (in $H^{-1}(\Omega)$) and $j \in L^2(\Omega)$ (in $[H_0^1(\Omega) \cap H^2(\Omega)]^*$). We denote (v, q) the associated solution to the Stokes-Brinkman problem (4)-(5). We have also at-hand a solution $(u^{(N)}, p^{(N)})$. Then **Theorem 1** entails that, then $u^{(N)} \to v$ in $L^2(\Omega)$. Note that we always consider that $u^{(N)}$ is extended by its boundary values on the particles. Actually, the minimization property of Stokes-solution ensures that $u^{(N)}$ is bounded in $H_0^1(\Omega)$ so that the convergence holds also in $H_0^1(\Omega) - w$.

4. The Darcy regime.

We proceed with the Darcy regime. We mean here that we consider a choice of radius for which there exists $r_0 > 0$ such that:

$$\frac{r_0}{N} \leqslant R \leqslant \frac{\lambda_{min}}{4N^{1/3}}. \tag{20}$$

We recall that λ_{min} is associated with assumption (H1). We recall also that, in this case, we introduce a target fluid flow $v \in L^2(\Omega)$ and the following functional description of the discrete set of particles and velocities V_i:

$$\tilde{v}_d = \sum_{i=1}^N \mathbb{1}_{\tilde{B}_i} V_i \,, \quad \tilde{\rho}_d = \sum_{i=1}^N \mathbb{1}_{\tilde{B}_i},$$

where $\tilde{B}_i = B_\infty(h_i, \lambda_{min} N^{-1/3}/2)$.

4.1. **Proof of Theorem 2.** The proof of **Theorem 2** splits into two parts. First, we show that the "good unknown" in this regime is the velocity

$$\tilde{u}(x) = \sqrt{\tilde{\rho}_d} u.$$

Indeed, we have the following lemma:

Lemma 6. *There exists a constant $C(\lambda_{min})$ such that:*

$$\|\nabla u\|_{L^2(\Omega)} \leqslant C(\lambda_{min}) \sqrt{RN} \|\mathbf{V}\|_{\ell^2},$$
$$\|\tilde{u}\|_{L^2(\Omega)} \leqslant C(\lambda_{min}) \|\mathbf{V}\|_{\ell^2}.$$

Proof. We prove this lemma by adapting **Lemma 4** and [2, Lemma 3.4.1]. Indeed, we remark first that we can construct the same u_0 as in **Lemma 4** which satisfies again $u_0 \in D(\Omega)$ with $u_0 = V_i$ on B_i, and

$$\|u_0\|_{L^2(\Omega)} \leqslant C(\lambda_{min}) \|\mathbf{V}\|_{\ell^2} \,, \qquad \|\nabla u_0\|_{L^2(\Omega)} \leqslant C(\lambda_{min}) \sqrt{RN} \|\mathbf{V}\|_{\ell^2}.$$

Consequently, via ever the same minimization principle, we use the second bound to yield that:

$$\|\nabla u\|_{L^2(\Omega)} \leqslant C(\lambda_{min}) \sqrt{RN} \|\mathbf{V}\|_{\ell^2}.$$

We can then consider $w = u - u_0 \in D(\Omega)$. Since $w = 0$ on B_i we can use a straightforward extension of [2, Lemma 3.4.1] to the case of spheres for the ℓ_∞-norm to yield that:

$$\|w\|^2_{L^2(\bigcup \tilde{B}_i)} \leqslant \frac{C}{RN} \|\nabla w\|^2_{L^2(\Omega)} \quad \forall\, i = 1, \ldots, N.$$

Combining this inequality with the previous bounds on u_0 and u we infer that:

$$\begin{aligned} \|w\|^2_{L^2(\bigcup \tilde{B}_i)} &\leqslant \frac{C}{RN} \left(\|\nabla u_0\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \right), \\ &\leqslant C \|\mathbf{V}\|^2_{\ell^2} \end{aligned}$$

Finally, we have:

$$\|\tilde{u}\|^2_{L^2(\Omega)} = \|u\|^2_{L^2(\bigcup \tilde{B}_i)} \leqslant C \left(\|u_0\|^2_{L^2(\Omega)} + \|w\|^2_{L^2(\bigcup \tilde{B}_i)} \right).$$

and we conclude by combining the previous bounds for w and u_0. □

We point out that the previous lemma only gives a $O(1)-$bound for $w = u - u_0$. But at this point we have not been able to construct an extension of the distribution of velocities V_i to which we can compare u (for large N for instance). This is a novelty in comparison with the homogeneous case considered in [2].

The main effort of the proof is now a novel interpretation of the identity (12) in **Lemma 3.** Namely, we remark that **Lemma 3** holds true in this new context since convention (20) implies condition (11). So, we fix $w \in H^3(\Omega) \cap D(\Omega)$ and apply **Lemma 3**. We obtain:

$$6\pi R \sum_{i=1}^{N} \bar{w}_i \left(V_i - \tilde{u}_i \right) = \int_\Omega \nabla u : \nabla w - \langle err, w \rangle.$$

With the explicit values for $\bar{w}_i$ and $\tilde{u}_i$ and with the controls of item i) for err and of **Lemma 6** for $\|\nabla u\|_{L^2(\Omega)}$ we derive:

$$\begin{aligned} (21) \quad & \left| 6\pi R \sum_{i=1}^{N} \frac{1}{|B_i|} \int_{B_i} V_i \cdot w(x) - 6\pi R \sum_{i=1}^{N} \frac{1}{|B_i|} \int_{B_i} w(x) \cdot \frac{1}{|\tilde{B}_i|} \int_{\tilde{B}_i} u(x) \right| \\ & \qquad \leqslant C(\Omega, \lambda_{min}) \left(RN^{2/3} + 1 \right) \sqrt{RN} \|\mathbf{V}\|_{\ell^2} \|w\|_{H^2(\Omega)}. \end{aligned}$$

Here we rewrite the left-hand side using that $w \in H^3(\Omega) \subset C^{0,1}(\bar{\Omega})$. This entails that, for any $i \in \{1, \ldots, N\}$

$$\left| \frac{1}{|B_i|} \int_{B_i} w(x) - \frac{1}{|\tilde{B}_i|} \int_{\tilde{B}_i} w(x) \right| \leqslant \frac{C(\lambda_{min}) \|w\|_{H^3(\Omega)}}{N^{1/3}}.$$

Consequently, noting that $|\tilde{B}_i| = \lambda_{min}^3/8N$ for all i, we can transform the first term in the left-hand side of (21) since we obtain

$$(22) \quad \left| 6\pi R \sum_{i=1}^{N} \frac{1}{|B_i|} \int_{B_i} V_i \cdot w(x) - \frac{48\pi RN}{\lambda_{min}^3} \int_\Omega \tilde{v}_d(x) \cdot w(x) \right| \\ \leqslant C(\lambda_{min}) RN^{2/3} \|w\|_{H^3(\Omega)} \|\mathbf{V}\|_{\ell^2}.$$

In our setting $\sqrt{RN}$ is large. So, the new induced error term on the right-hand side of this latter inequality is smaller than:

$$C(\lambda_{min}, r_0) RN^{2/3} \sqrt{RN} \|w\|_{H^3(\Omega)} \|\mathbf{V}\|_{\ell^2},$$

corresponding to a similar right-hand side as for (21) in terms of powers of N. As for the other term of the left-hand side of (21), we use again that $H^3(\Omega) \subset C^{0,1}(\bar{\Omega})$ to yield that, for any $i \in \{1, \ldots, N\}$, there holds:

$$\left| \frac{1}{|B_i|} \int_{B_i} w(x) \cdot \int_{\tilde{B}_i} u(x) - \int_{\tilde{B}_i} w(x) \cdot u(x) \right| \leqslant \frac{1}{N^{1/3}} \left(\int_{\tilde{B}_i} |\tilde{u}(x)| \right) \|w\|_{H^3(\Omega)}.$$

Summing over i enables to transform the second term in the left-hand side of (21) since, combining the previous computation with the bound of **Lemma 6** for $\|\tilde{u}\|_{L^2(\Omega)}$, we obtain:

$$(23) \quad \left| 6\pi R \sum_{i=1}^{N} \frac{1}{|B_i|} \int_{B_i} w(x) \cdot \frac{1}{|\tilde{B}_i|} \int_{\tilde{B}_i} u(x) - \frac{48\pi RN}{\lambda_{min}^3} \int_\Omega w(x) \cdot \tilde{u}(x) \right| \\ \leqslant C(\lambda_{min}) \frac{RN}{N^{1/3}} \|\mathbf{V}\|_{\ell^2} \|w\|_{H^2(\Omega)}.$$

Eventually, combining (21) with (22)-(23), we obtain:

$$\left| \int_\Omega \tilde{v}_d(x) \cdot w(x) - \int_\Omega w(x) \cdot \tilde{u}(x) \right| \leqslant C(\Omega, \lambda_{min}, r_0) \left(\frac{\sqrt{RN}}{N^{1/3}} + \frac{1}{\sqrt{RN}} \right) \|\mathbf{V}\|_{\ell^2} \|w\|_{H^3(\Omega)}.$$

This concludes the proof.

4.2. **Comments.** As in the Brinkman regime, the results we propose above enable to tackle the issue $N \to \infty$ in the problem (1)-(2). Precisely, assume again that we choose as parameter the number N of holes in Ω and consider a sequence of configuration $(\mathbf{h}^{(N)}, R^{(N)})$ where $\mathbf{h}^{(N)} \in \mathcal{C}$ contains N centers and $R^{(N)} = r_{sc}/N^\alpha$ for some constant $r_{sc} > 0$ and exponent $\alpha > 0$. We also consider a sequence $\mathbf{V}^{(N)}$ of boundary data. To this sequence of boundary data we associate the discrete momentums and densities:

$$\tilde{\rho}_d^{(N)} := \sum_{i=1}^{N} \mathbb{1}_{\tilde{B}_i^{(N)}}, \qquad \tilde{v}_d^{(N)} := \sum_{i=1}^{N} \mathbb{1}_{\tilde{B}_i^{(N)}} V_i^{(N)}.$$

with obvious notations. Assume further that

(A1) r_{sc} is arbitrary and $\alpha \in (1/3, 1)$

(A2) there exists a constant M such that $\|\mathbf{V}^{(N)}\|_{\ell^2} \leqslant M$ for all N.

With (A2), we note that $\tilde{v}_d^{(N)}$ is bounded in $L^2(\Omega)$ and that both $\tilde{v}_d^{(N)}$ and $\tilde{\rho}_d^{(N)}$ have support in K. We can then assume that $\tilde{v}_d^{(N)}$ converges to some $v \in L^2(\Omega)$ in $[H^3(\Omega) \cap D(\Omega)]^*$. Furthermore, we have also at-hand the solution $(u^{(N)}, p^{(N)})$ to (1)-(2) and

$$\tilde{u}^{(N)} = \sqrt{\tilde{\rho}_d^{(N)}} u^{(N)}.$$

Then, **Lemma 6** entails that, $\tilde{u}^{(N)}$ is bounded in $L^2(\Omega)$ and **Theorem 2** ensures that it converges (in $L^2(\Omega) - w$) to v up to the addition of a gradient (since the convergence holds in the dual of $D(\Omega)$). We recall that, if the B_i are distributed periodically and well-centered in a cube K, we have $\sqrt{\tilde{\rho}_d^{(N)}} = \mathbb{1}_K$ whatever the value of N up to choosing the optimal value for λ_{min}. In this case $\tilde{u}^{(N)}$ is the restriction to K of $u^{(N)}$.

Declarations.
Funding. MH acknowledges support from Institut Universitaire de France and SingFlows project (ANR-18- CE40-0027) of the French National Research Agency (ANR).

Conflict of interests. The author declares he has no conflict of interest.

References

[1] G. Allaire. Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes. I: Abstract framework, a volume distribution of holes. *Arch. Rational Mech. Anal.*, 113(3):209-259, 1991.
[2] G. Allaire. Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes. II: Non critical size of the holes for a volume distribution and a surface distribution of holes *Arch. Rational Mech. Anal.*, 113(3):261-298, 1991.
[3] H. C. Brinkman. A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles. *Applied Scientific Research*, A.1:27-34, 1947.
[4] K. Carrapatoso and M. Hillairet. On the derivation of a Stokes-Brinkman problem from Stokes equations around a random array of moving spheres. *Comm. Math. Phys.* 373:265–325, 2020.
[5] L. Desvillettes, F. Golse, and V. Ricci. The mean-field limit for solid particles in a Navier-Stokes flow. *J. Stat. Phys.*, 131(5):941-967, 2008.
[6] D. Gérard-Varet. A simple justification of effective models for conducting or fluid media with dilute spherical inclusions arXiv:1909.11931, March 2020.
[7] D. Gérard-Varet and M. Hillairet. Analysis of the viscosity of dilute suspensions beyond Einstein's formula. *Arch. Ration. Mech. Anal.* 238:1349–1411, 2020.
[8] D. Gérard-Varet and R. Höfer. Mild assumptions for the derivation of Einstein's effective viscosity formula arXiv:2002.04846, November 2020.
[9] D. Gérard-Varet and A. Mecherbet. On the correction to Einstein's formula for the effective viscosity arXiv:2004.05601, April 2020.
[10] M. Duerinckx and A. Gloria. On Einstein's effective viscosity formula arXiv:2008.03837, September 2020.
[11] A. Giunti. Derivation of Darcy's law in randomly punctured domains arXiv:2101.01046, January 2021.
[12] M. Hillairet. On the homogenization of the stokes problem in a perforated domain. *Arch. Ration. Mech. Anal.* 230:1179-1228, 2018.

[13] M. Hillairet and D. Wu. Effective viscosity of a polydispersed suspension. *J. Math. Pures Appl.* 138:413–447, 2020.
[14] M. Hillairet, A. Moussa and F. Sueur. Computation of the Brinkman force induced by a polydispersed cloud of particles. *Kinet. Relat. Models* 12:681–701, 2019.
[15] R. Höfer, and J.J.L. Velàzquez. The method of reflections, homogenization and screening for poisson and stokes equations in perforated domains. arXiv:1603.06750, March 2016.
[16] A. Mecherbet and M. Hillairet. L^p estimates for the homogenization of the stokes problem in a perforated domain. *J. Inst. Math. Jussieu* 19:231–258, 2020.

Institut Montpelliérain Alexander Grothendieck, Université de Montpellier, Place Eugène Bataillon, 34095 Montpellier Cedex 5 France
Email address: `matthieu.hillairet@umontpellier.fr`